Math 366, Final Exam

This is a 100 point exam.

1. (20 points) (a) State PRECISELY the Arzela-Ascoli theorem.

(b) State PRECISELY the Inverse Function Theorem.

(c) Define contraction mapping and state the contraction mapping theorem.

2. (15 points) Let M_{mn} denote the collection of all $m \times n$ matrices.

(1) Define two different norms in M_{mn}

(2) Compare these two norms for any element in M_{mn}.

3. (20 points) Let X be a metric space and M a subset in X.

(1) Define precisely that M is compact in X.

(2) Prove by the definition that if M is compact then M is closed.

(3) Suppose M is compact and $F \subset M$ is closed. Prove that F is compact.

4. (10 points) Suppose that X is metric space and that $\{f_n\}$ is uniformly convergent sequence of bounded functions on X. Is it true that $\{f_n\}$ is uniformly bounded? Prove or give a counter example.

5. (15 points) Consider

$$f_1(x_1, x_2, x_3, y_1, y_2) = e^{y_1} + x_2 y_2 - 9x_1 x_3 - 1 = 0 f_2(x_1, x_2, x_3, y_1, y_2) = 5y_2 g(y_1) + g(y_1 + 2) + 2x_2 + 8x_3 = 0,$$

where $g(y_1)$ is continuously differentiable. We want to solve (y_1, y_2) in terms of (x_1, x_2, x_3) near $(0, 0, 0, 0, 0)$. Find the condition(s) on $g(y_1)$ so that the implicit function theorem can be used near $(0, 0, 0, 0, 0)$.

6. (20 points) Consider the family of continuous functions

$$f_n(x,y) = \frac{x^2 + y^2 + 5xy}{x^2 + y^2 + (\frac{1}{n})^2}$$

on the disk $B_1 = \{(x,y); x^2 + y^2 \leq 1\}$.

(1) Find the pointwise limit $f(x,y)$ of $f_n(x,y)$ on B_1.

(2) Show that f is not continuous.

(3) Is the convergence $f_n(x,y) \to f(x,y)$ uniform on B_1? Justify your result.