Math 366, Midterm Exam

This is a 100 point exam.

1. (20 points) True of False. (No proof or counter example needed).

(a) The limit function of a sequence of continuous functions is always continuous.

(b) If $\{f_n(x)\}$ and $\{g_n(x)\}$ converge uniformly on a set E, then their sum $\{f_n(x) + g(x)\}$ also converges uniformly on E.

(c) A differential function is always equal to its Taylor expansion in the interval of convergence.

(d) A continuous function in an interval is differentiable at least at one point.

2. (20 points) (a) Define PRECISELY the equicontinuity for a sequence of functions in E.

(b) Define PRECISELY the Fourier series of a 2π periodic function.

(c) State PRECISELY the Weierstrass Approximation Theorem.

(d) State PRECISELY the Bessel's inequality.

3. (10 points) Let

$$f_n(x) = \frac{x^2}{1 + nx}.$$

Is $\{f_n(x)\}$ convergent uniformly in $[0, 1]$? Verify your answer.

4. (10 points) Suppose that $\{f_n(x)\}$ is a sequence of uniformly bounded Riemann integrable functions on $[0, 1]$. Let

$$F_n(x) = \int_0^x f_n(t)dt \quad 0 \leq x \leq 1.$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly in $[0, 1]$.

5. (10 points) Calculate the following limit:

$$\lim_{x \to 0} (\frac{1+e^x}{2})^{\frac{1}{x}}.$$

6. (15 points) Calculate the Fourier series of the function $f(x) = x^2$ in $(-\pi, \pi)$ and then find the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

7. (15 points) (a) Suppose f is continuous on $[0,1]$ and

$$\int_0^1 f(x)x^n dx = 0, \quad n = 0, 1, 2, \cdots.$$

Prove that $f(x) \equiv 0$ in $[0,1]$.

(b) Suppose g is continuous on $[0,1]$ and

$$\int_0^1 g(x)x^n dx = 0, \quad n = 1998, 1999, 2000, \cdots.$$

Prove that $g(x) \equiv 0$ in $[0,1]$.