METRIC SPACES

In this note we discuss various properties of metric spaces.

§1. Metric Spaces

A metric space is a topological space where the topology is determined by a distance function.

Definition. A metric space $(,d)$ is a nonempty set with a real-valued function d defined on $\times$ such that for any $x, y, z \in$

(a) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, y) \leq d(x, z) + d(z, y)$.

The function d is called a metric.

The purpose of introducing metrics is to describe convergence.

Definition. A sequence $\{x_n\}$ in metric space $(,d)$ is convergent to $x \in$ if $\lim_{n\to\infty} d(x_n, x) = 0$. A sequence $\{x_n\}$ is Cauchy if $\lim_{m,n\to\infty} d(x_n, x_m) = 0$.

Definition. A metric space $(,d)$ is complete if any Cauchy sequence is convergent.

Remark A metric space is the generalization of the Euclidean space. Let $\mathbb{R}$ denote the collection of all real numbers and set $d(x, y) = |x - y|$ for any $x, y \in \mathbb{R}$. Then $(\mathbb{R}, d)$ is a complete metric space.

Example Let $C[a, b]$ denote the collection of all continuous functions in $[a, b] \subset \mathbb{R}$. Set

$$d(x, y) = \max_{a\leq t\leq b} |x(t) - y(t)| \quad \text{for any } x, y \in C[a, b].$$

Then $(C[a, b], d)$ is a metric space. In $C[a, b]$, a sequence $\{x_n\}$ convergent to x is equivalent to the fact that $\{x_n(t)\}$ is uniformly convergent to $x(t)$ in $[a, b]$. In fact $(C[a, b], d)$ is complete.

Now we use metric to define a topology. In the following we always assume that $(,d)$ is a metric space.

Definition. A set $O \subset$ is open if for any $x \in O$ there exists an $r > 0$ such that $y \in O$ for any $y \in$ with $d(x, y) < r$.

Proposition The set and ϕ are open; the intersection of any two open sets is open; and the union of any collection of open sets is open.

Definition. A set $F \subset$ is closed if F^c is open.

Proposition A set $F \subset$ is closed if and only if for any sequence $\{x_n\} \subset F$ which is convergent in the limit x belongs to F.

Suppose $(, d)$ and $(, \rho)$ are two metric spaces.

Definition. A mapping $A :\to$ is continuous at $x_0 \in$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x, x_0) < \delta$ implies $\rho(A(x), A(x_0)) < \varepsilon$.

Proposition A mapping $A :\to$ is continuous at $x_0 \in$ if and only if for any open set $O \subset$ containing $A(x_0)$ the set $A^{-1}(O)$ is open .

§2. Compactness

We always assume that $(, d)$ is a metric space. For any x_0 and $r > 0$ we denote

$$B(x_0; r) = \{x \in ; d(x, x_0) < r\}.$$

A subset $A \subset$ is bounded if $A \subset B(x_0; r)$ for some x_0 and $r > 0$.

In $\mathbb{R}^n$ any bounded infinite set contains a convergent subsequence. This is not true in the general metric space.

Example In $C[0, 1]$ consider the sequence

$$x_n(t) = \{\, 0 \; fort \geq \frac{1}{n} 1 - nt fort \leq \frac{1}{n}$$

for $n = 1, 2, \cdots$. Obviously $\{x_n\} \subset B(0; 1)$ where 0 denotes the identically zero function. However $\{x_n\}$ has no convergent subsequence in $C[0, 1]$.

Definition. In a metric space $(, d)$, a subset M is compact if any open covering of M has a finite subcovering.

Theorem In a metric space $(, d)$, a subset M is compact if and only if M is closed and any sequence in M has a convergent subsequence. Proof $\Rightarrow$ Suppose M is compact. First we prove that M^c is open. For any $x_0 \in M^c$ we have $M \subset \cup_{x \in M} B(x; d(x, x_0)/2)$. By compactness there exist $x_1, \cdots, x_n \in M$ such that $M \subset \cup_{i=1}^{n} B(x_i; d(x_i, x_0)/2)$. Take $\delta = \min_{1 \leq i \leq n} d(x_i, x_0)/2 > 0$. Then $M \cap B(x_0, \delta) = \phi$. Hence M^c is open.

Next we take any sequence $\{x_n\} \subset M$. We prove by contradiction that $\{x_n\}$ has a convergent subsequence. Suppose the sequence $\{x_n\} \subset M$ has no convergent subsequences. We may assume $x_n \neq x_m$ for $n \neq m$. For any $n \in \mathbb{N}$ the set $S_n = \{x_1, \cdots, x_{n-1}, x_{n+1}, \cdots\}$ is closed since it does not have convergent subsequence. Hence $\backslash S_n$ is open. Note $\cup_{n=1}^{\infty}(\backslash S_n) = \backslash \cap_{n=1}^{\infty} S_n = \supset M$. By the compactness there exists an $N \in \mathbb{N}$ such that $\cup_{n=1}^{N} \backslash S_n \supset M$, i.e., $\backslash\{x_{N+1}, x_{N+2}, \cdots\} \supset M$. This is impossible since $x_{N+1} \subset M$ while $x_{N+1} \notin \backslash\{x_{N+1}, x_{N+2}, \cdots\}$. Contradiction.

$\Leftarrow$ Suppose M is closed and that any sequence in M has a convergent subsequence. We first prove that for any $\varepsilon > 0$ there exist finitely many points $x_1, \cdots, x_k$ in M for some $k = k(\varepsilon)$ such that $M \subset \cup_{i=1}^{k} B(x_i; \varepsilon)$.

If it is not true, then there exists an $\varepsilon_0 > 0$ satifying the following property:

for some $x_1 \in M$ there exists an $x_2 \in M \backslash B(x_1; \varepsilon_0)$;

for $\{x_1, x_2\} \subset M$ there exists an $x_3 \in M \backslash B(x_1; \varepsilon_0) \cup B(x_2; \varepsilon_0)$;

$\cdots\cdots$

for $\{x_1, \cdots, x_n\} \subset M$ there exists an $x_{n+1} \in M \backslash \cup_{i=1}^n B(x_i; \varepsilon_0)$;

$\cdots\cdots$.

The sequence $\{x_n\} \subset M$ satisfies $d(x_n, x_m) \geq \varepsilon_0$ for $n \neq m$. It could not have convergent subsequence. This is a contradiction.

Take any open covering $\{G_\lambda\}$ of M. Suppose it has no finite subcovering. For any $n \in \mathbb{N}$ there exist $x_1^{(n)}, \cdots, x_{k(n)}^{(n)} \in M$ such that $M \subset \cup_{i=1}^{k(n)} B(x_i(n); 1/n)$. Hence for any $n \in \mathbb{N}$ there exists a $y_n \in \{x_1^{(n)}, \cdots, x_{k(n)}^{(n)}\} \subset M$ such that $B(y_n; 1/n)$ cannot be covered by finitely many G_λ. By assumptions there exists a convergent subsequence $\{y_{n_k}\}$ with limit $y \in M$. We may assume $y \in G_{\lambda_0}$. Since G_{λ_0} is open there exists a $\delta > 0$ such that $B(y; \delta) \subset G_{\lambda_0}$. For such δ we may take k large such that $B(y_{n_k}; 1/n_k) \subset B(y, \delta) \subset G_{\lambda_0}$. This contradicts the assumption that every $B(y_n; 1/n)$ cannot be covered by finitely many G_λ.

§3. Continuous Mappings on Compact Spaces

Next we discuss properties of continuous mappings on compact metric space. We state some results without proof.

Theorem Suppose A is a continuous mapping of a compact metric space into a metric space . Then $f()$ is compact.

Corollary Suppose f is a continuous function in a compact metric space . Set

$$M = \sup_{x\in} f(x), \quad m = \inf_{x\in} f(x).$$

Then there exist points $p, q \in$ such that $f(p) = M$ and $f(q) = m$.

Definition. Suppose A is a mapping: $\to$. A is uniformly continuous in if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\rho(A(x_1), A(x_2)) < \varepsilon \quad if\ d(x_1, x_2) < \delta,\ and\ x_1, x_2 \in .$$

Theorem Suppose A is a continuous mapping from a compact metric space into a metric space . Then A is uniformly continuous.

Next we generalize the notion of continuous functions in bounded closed intervals. Suppose (M, ρ) is a compact metric space. $C(M)$ denotes the collection of all continuous functions: $M \to \mathbb{R}$. Define

$$d(u, v) = \max_{x \in M} |u(x) - v(x)| \quad \text{for any } u, v \in C(M).$$

Lemma $(C(M), d)$ is a complete metric space. Moreover the convergence in $(C(M), d)$ is equivalent to the uniform convergence.

Definition. Suppose $\{f_n\}$ is a sequence in $C(M)$.

(a) $\{f_n\}$ is uniformly bounded if for some $M > 0$ there holds $|f_n(x)| \leq M$ for any $x \in M$ and any integer n;

(b) $\{f_n\}$ is equicontinuous if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|f_n(x_1) - f_n(x_2)| < \varepsilon$ for any integer n and $x_1, x_2 \in M$ with $\rho(x_1, x_2) < \delta$.

Arzela-Ascoli Theorem Suppose $\{f_n\}$ is a sequence in $C(M)$. Then $\{f_n\}$ has a convergent subsequence (with respect to the metric d defined above) if and only if $\{f_n\}$ is uniformly bounded and equicontinuous.

§4. Contraction Mapping Theorem

Suppose $(, d)$ is a metric space and $A :\to$ is a mapping. In lots of cases we need to investigate the solvability of $Ax = x$ in , i.e., whether A has a fixed point in .

Definition. The mapping $A : (, d) \to (, d)$ is a contraction mapping if for some $\alpha \in (0, 1)$ there holds $d(Ax, Ay) \leq \alpha d(x, y)$ for any $x, y \in$.

Remark Contraction mappings are always continuous.

The following theorem is the simplest and most widely used existence result in functional analysis.

Contraction Mapping Theorem Suppose $(, d)$ is a complete metric space and $A : (, d) \to (, d)$ is a contraction mapping. Then A has a unique fixed point on .

Proof For any $x_0 \in$ set $x_{n+1} = Ax_n$ for $n = 0, 1, 2, \cdots$. We claim that $\{x_n\}$ is a Cauchy sequence. To see this we calculate

$$d(x_{n+1}, x_n) = d(Ax_n, Ax_{n-1}) \leq \alpha d(x_n, x_{n-1}) \leq \cdots \leq \alpha^n d(x_1, x_0).$$

Hence for any $n, p \in \mathbb{N}$, there holds

$$d(x_{n+p}, x_n) \leq \sum_{i=1}^{p} d(x_{n+i}, x_{n+i-1}) \leq \sum_{i=1}^{p} \alpha^{n+i-1} d(x_1, x_0) \leq \sum_{i=1}^{\infty} \alpha^{n+i-1} d(x_1, x_0) = \frac{\alpha^n}{1 - \alpha} d(x_1, x_0).$$

Hence $\{x_n\}$ is a Cauchy sequence, and convergent to x^* in . Taking the limit in $Ax_n = x_{n+1}$ we conclude $Ax^* = x^*$ by continuity of A.

If x^{**} also satisfies $Ax^{**} = x^{**}$, then $d(x^*, x^{**}) = d(Ax^*, Ax^{**}) \leq \alpha d(x^*, x^{**})$. This implies $x^* = x^{**}$.

As an application we discuss the initial value problem for the following ordinary differential equation

$$\frac{dx}{dt} = f(t, x), \quad x(0) = \xi 1$$

where f is a continuous function in $[-a, a] \times [\xi - b, \xi + b]$ for some $a, b \in (0, \infty)$. We consider its equivalent integral equation

$$x(t) = \xi + \int_0^t f(s, x(s))ds$$

and view it as a fixed point problem.

For some $h \in (0, a]$, consider

$$= \{x \in C[-h, h];\ \max_{-h \leq t \leq h} |x(t) - \xi| \leq b\}.$$

is a closed subset in $C[-h, h]$ and hence a complete metric space. Set

$$Ax(t) = \xi + \int_0^t f(s, x(s))ds \quad \text{for any } x \in .$$

Obviously $Ax \in C[-h, h]$. First we need to check that $Ax \in$ for any $x \in$. To do this we have for $|t| \leq h$

$$|Ax(t) - \xi| = |\int_0^t f(s, x(s))ds| \leq \int_0^h |f(s, x(s))|ds.$$

We assume that

$$M = \sup_{|t| \leq a, |x - \xi| \leq b} |f(t, x)|.2$$

Then

$$\max_{|t| \leq h} |Ax(t) - \xi| \leq hM.$$

Hence $Ax \in$ for any $x \in$ if we choose $h \leq a$ such that $hM \leq b$. Next for any $x, y \in$ we consider

$$Ax(t) - Ay(t) = \int_0^t f(s, x(s))ds - \int_0^t f(s, y(s))ds.$$

We assume for some positive constant L

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|3$$

for any $|t| \leq a$ and $|x_1 - \xi| \leq b$, $|x_2 - \xi| \leq b$. Then

$$\max_{|t| \leq h} |Ax(t) - Ay(t)| \leq h \max_{|t| \leq h} |f(t, x(t)) - f(t, y(t))| \leq hL \max_{|t| \leq h} |x(t) - y(t)|.$$

Hence A is a contraction mapping if $hL < 1$. We may apply the Contraction Mapping Theorem if we choose

$$h \leq a, \quad h \leq \frac{b}{M}, \quad h < \frac{1}{L}.4$$

Theorem The initial value problem (1) has a unique solution $x(t)$ in $[-h, h]$ for h in (4) if $f \in C([-a, a] \times [\xi - b, \xi + b])$ satisfies (2) and (3).