Math 366: Honors Analysis II Name:__________________________

Final Exam *May 11, 2001*

1. (30 pts) Give precise definitions for the following concepts.

a) uniform convergence

b) convolution

c) metric space

d) compact set

e) complete set

f) connected set and arc-wise connected set

g) Borel sets

h) Lebesgue measure

i) measurable function

j) Lebesgue integral

2. (50 pts) Give precise statements of the following theorems.

a) Weierstrass Approximation Theorem

b) Arzela-Ascoli Theorem

c) Contractive Mapping Principle

d) Implicit Function Theorem

e) Lebesgue Monotone Convergence Theorem

3. (10 pts) Prove that A is open if and only if A^c is closed.

4. (10 pts) Prove that if M is a compact metric space and $f : M \to$ is continuous, then f is uniformly continuous.

5. (10 pts) Prove that if f is analytic on (a, b) then the values of f on (a, b) are determined by its values on any subinterval $(c, d) \subset (a, b)$.

6. (10 pts) Let M be the set of continuous functions $f : [0,1] \to [0,1]$. Prove that M with the sup-norm is a complete metric space.

7. (10 pts) Let M be as in problem 6 and consider the transformation

$$Tx(t) = \frac{1}{2} + \frac{1}{4}\int_0^t x(s)(1 - x(s))\,ds$$

Prove that $T : M \to M$ and is a contractive mapping. Then find the fixed point of T.

8. (10 pts) Determine which level curves of $F(x, y) = x^2 + y^2 - x^4 - y^4$ have singular points and where those singular points occur. Then describe how the level curves $F(x, y) = c$ vary as c varies.

9. (10 pts) Define the Cantor set. Prove that it is uncountable and has measure 0.

10. (10 pts) Prove the following statements about measurable sets with respect to a measure μ.

1. If $A \subset B$ then $\mu(A) \leq \mu(B)$.
2. If $A_1 \subset A_2 \subset A_3 \ldots$, then $\mu(\bigcup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mu(A_j)$.

11. (10 pts) Prove the following statements about Lebesgue integrals of measurable functions on measurable sets.

1. If $\mu(A) = 0$ and $f \in_\mu (A)$ then $\int_A f \, d\mu = 0$.
2. If $f, g \in_\mu (A)$ and $f = g$ almost everywhere on A then $\int_A f \, d\mu = \int_A g \, d\mu$.

12. (10 pts) Define Hausdorff measure, μ^α, on Borel sets. Prove that that μ^0 is counting measure and μ^1 is Lebesgue measure.