1 Basic Concepts

1.1 Limit, Supremum, Infimum

Definition 1 *x is the* limit *if x_n as $n \to \infty$, $\lim_{n\to\infty} x_n = x$, if $\forall\, 1/m$, $\exists\, N$ such that $n \geq N \Rightarrow |x - x_n| < 1/m$.*

Definition 2 *If $E \subset$ is bounded above, then there exists a unique real number,* $\sup E$*, called the* supremum *of E, such that*

1. $\sup E$ *is an upper bound of E.*
2. *If y is any upper bound of E, then $y \geq \sup E$.*

Definition 3 *If $E \subset$ is bounded below, then there exists a unique real number,* $\inf E$*, called the* infimum *of E, such that*

1. $\inf E$ *is a lower bound of E.*
2. *If y is any lower bound of E, then $y \leq \sup E$.*

Definition 4 *$\{y_n\}$ is a* subsequence *of $\{x_n\}$ if $\exists\, m :\to$ such that $y_n = x_{m(n)}$, $\forall\, n$.*

We sometimes denote a subsequence by $\{x_{n'}\}$ where n' stands for $m(n)$.

Definition 5 *x is a* limit point *of $\{x_n\}$ if there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $\lim_{n'\to\infty} x_{n'} = x$.*

Example: $x_n = \begin{cases} 1 & n \text{ even} \\ 1/n & n \text{ odd} \end{cases}$

0 and 1 are limit points, but neither is the limit of x_n.

Note: A convergent sequence has only one limit point.

Definition 6 *The* limsup *of a sequence $\{x_n\}$ is the limit of the sequence $y_k = \sup_{n\geq k}\{x_n\}$,*

$$\limsup\{x_n\} = \lim_{k\to\infty} y_k = \lim_{k\to\infty} \sup_{n\geq k}\{x_n\}$$

Definition 7 *The* liminf *of a sequence* $\{x_n\}$ *is the limit of the sequence* $z_k = \inf_{n \geq k}\{x_n\}$*,*

$$\liminf\{x_n\} = \lim_{k\to\infty} z_k = \lim_{k\to\infty} \inf_{n\geq k}\{x_n\}$$

The sequence y_k is decreasing, $y_{k+1} \leq y_k$, and the sequence z_k is increasing, $z_{k+1} \geq z_k$. Therefore, if we allow limits to be $\pm\infty$, $\limsup\{x_n\}$ and $\liminf\{x_n\}$ always exist. Note also that $\limsup\{x_n\}$ is the *maximum* of all limit points, while $\liminf\{x_n\}$ is the *minimum* of all limit points.

Example: As above, $\limsup\{x_n\} = 1$, $\liminf\{x_n\} = 0$.

1.2 Open Sets

Definition 8 *$A \subset$ is* open *if $\forall\, x \in A$, there is an interval $I_x = (a, b)$ such that $x \in I_x \subset A$.*

Example: $A = \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right)$

Theorem 1 *$A \subset$ is open $\iff$ A is a countable disjoint union of intervals.*

$(\Rightarrow)$ If A is open there is an interval $I_x \subset A$ around each of its points $x \in A$. Thus $A = \bigcup_{x\in A} I_x$.

If two intervals $I_x = (a, d)$, $I_y = (c, d)$ are not disjoint, then either $I_x \cup I_y = (a, d)$ or $I_x \cup I_y = (a, b)$ so $I_x \cup I_y$ can be combined into one interval. Therefore, A is the union of *disjoint* open intervals.

How many? Each disjoint interval contains a (different) rational number, and is countable, so at most a *countable number.*

The converse $(\Leftarrow)$ is obvious.

Theorem 2 *1. The union of any number of open sets is open.*

2. The intersection of a finite number of open sets is open.

1. Let $x \in \bigcup U_\alpha$ where each U_α is open. Then $x \in U_\alpha$ for some α so there is an interval (a, b) in U_α containing x. Then $x \in (a, b) \subset \bigcup U_\alpha$, proving that the union is open.

2. Let $x \in U_1 \cap \ldots \cap U_n$ where each U_i is open. For each i there is an interval (a_i, b_i) containing x so $x \in (a_1, b_1) \cap \ldots \cap (a_n, b_n) \subset U_1 \cap \ldots \cap U_n$. This proves the intersection $U_1 \cap \ldots \cap U_n$ is open since $(a_1, b_1) \cap \ldots \cap (a_n, b_n) = (a, b)$ where $a = \max\{a_i\}$ and $b = \min\{b_i\}$.

Note: $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$, so the assumption of finite is necessary in 2.

1.3 Closed Sets

Definition 9 *We say that x is a* limit point *of a set $A \subset$ if $\exists\{x_n\} \subset A$ such that $x = \lim x_n$.*

Note that an equivalent condition for x to be a limit point of A is that every open set containing x contains a point of A.

Definition 10 *A set $A \subset$ is* closed *if it contains all its limit points.*

Examples: $[a, b]$, $(-\infty, \infty)$ (also open!), $\emptyset$, any finite set.

Theorem 3 *A is closed $\iff$ A^c is open.*

($\Rightarrow$) Let $y \in A^c$. Then y is not a limit point of A (closed). So there is an open interval I_y, $y \in I_y \subset A^c$ (otherwise every open interval containing y intersects A which implies y is a limit point of A). $\Rightarrow A^c$ is open.

($\Leftarrow$) Let x be a limit point of A. If $x \in A^c$ then $\exists\, I_x$ such that $x \in I_x \subset A^c$ (open). Contradiction! Therefore, $x \in A$.

Using the above characterization of open sets, we see that a closed set is the complement of a countable union of disjoint open sets.

Theorem 4 1. *The union of a finite number of closed sets is closed.*

2. *The intersection of any number of closed sets is closed.*

These statements follow from Theorems **??**, **??**, and the following identities:

$$(U_1 \cup \ldots \cup U_n)^c = U_1^c \cap \ldots \cap U_n^c \text{ and } (\bigcap U_\alpha)^c = \bigcup U_\alpha^c$$

Note: $\bigcup_{n=1}^{\infty} [-1 + \frac{1}{n}, 1 - \frac{1}{n}] = (-1, 1)$, so the assumption of finite is necessary in 1.

Example (Cantor Set): Remove the middle third, $(1/3, 2/3)$, from $[0, 1]$ to get $[0, 1/3] \cup [2/3, 1]$. Next remove the middle third from the intervals $[0, 1/3]$ and $[2/3, 1]$ to get $[0, 1/9] \cup [2/9, 1/3] \cup [2/3, 2/9] \cup [4/9, 1]$. Repeat this process recursively with each closed subinterval. The resulting set C is called the *Cantor Set*. It contains all of its limit points (in fact, every point is a limit point), but C contains no intervals! It is also uncountable.

An equivalent way to define C is the set of numbers in $[0, 1]$ whose ternary expansion contains only 0's and 2's.

Definition 11 *Let $A \subset$. The* closure *of A, $\overline{A}$, is the union of A and all the limit points of A.*

Note: It is not hard to prove that $\overline{A}$ is the smallest closed set containing A (or, equivalently, that $\overline{A}$ is the intersection of all closed sets containing A). In fact, this statement could be used as the definition of $\overline{A}$.

Definition 12 *A subset $B \subset A$ is* dense *in A, if $A \subset \overline{B}$.*

Note: B is dense in A if every point of A is the limit of a sequence of points in B.

Examples: 1) is dense in .

2) (a, b) is dense in $[a, b]$.

3) $(0, 1) \setminus \{1/n \mid n \in\}$ is dense in $(0, 1)$.

1.4 Compact Sets

Definition 13 *A set $A \subset$ is* compact *if every open cover of A has a finite subcover:*

$$A \subset \bigcup_{x \in C} U_\alpha \Rightarrow A \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_N} \text{ for some } \alpha_1, \ldots, \alpha_N \in C$$

Here C is an indexing set and the U_α are open sets.

Theorem 5 *The following are equivalent:*

1. *A is compact*
2. *A is closed and bounded*
3. *Every sequence in A has a limit point in A.*

$1 \Rightarrow 2$: Let y be a limit point of A and suppose $y \notin A$. The complements of $[y - 1/n, y - 1/n]$ are open and cover $\setminus\{y\} \supset A$. Since A is compact, a finite subset of them cover A. But the sets are nested, so this means $\exists N$ such that A is contained in the complement of $[y - 1/N, y + 1/N]$. But this implies that y is not a limit point of A, a contradiction. Therefore, $y \in A$, proving that A is closed. To prove that A is bounded, consider the open cover $\{(x - 1, x + 1) \mid x \in A\}$. Since A is compact, $A \subset (x_1 - 1, x_1 + 1) \cup \cdots \cup (x_n - 1, x_n + 1)$ for some $x_1, \ldots, x_n \in A$. So A is bounded above by $\max\{x_i + 1\}$ and below by $\min\{x_i - 1\}$.

$2 \Rightarrow 3$: Let $\{x_n\} \subset A$. Since A is bounded, $y = \limsup\{x_n\}$ exists and is a limit point of $\{x_n\}$ and A. Since A is closed $y \in A$.

$3 \Rightarrow 1$: Let $= \{U_\alpha\}$ be an open cover of $A \subset \bigcup U_\alpha$. First we find a countable subset of that still covers A as follows. Let I_j, $j = 1, 2, 3, \ldots$, be the countable collection of intervals that have rational endpoints. For $j = 1, 2, 3, \ldots$ choose one $U_\beta \in$ that contains I_j, if any. Let $' = \{U_\beta\} \subset$ be the resulting countable subcollection. To see that $'$ still covers A, note that $x \in A \Rightarrow x \in U_\alpha \in$ for some α. Since U_α is open, it contains an interval around x, say $x \in (a, b) \subset U_\alpha$. By shrinking this interval, if necessary, we may assume $a, b \in$ and hence $(a, b) = I_j$ for some j. Therefore $\exists\, U_\beta \in'$ such that $x \in I_j \subset U_\beta$ and hence $'$ covers A.

We now have a countable subcover, say, $U_1, U_2, \ldots$. If we take n large enough, then $U_1, U_2, \ldots, U_n$ must already cover A. Suppose not. Then for each n, $\exists\, x_n \in A$ that is not contained in $U_1, \ldots, U_n$. By assumption, the sequence $\{x_n\}$ has a limit point $x \in A$. Thus $x \in U_k$ for some k. But by construction, U_k does not contain $x_k, x_{k+1}, \ldots$ contradicting the fact that any neighborhood of a limit point must contain an infinite number of points in the sequence.

Example: Let $a, b \in$. Then $[a, b]$ is closed and bounded, and so is compact. On the other hand, (a, b) is bounded but not closed, so it is not compact. The intervals, $(-\infty, b]$ and $[a, \infty)$ are closed, but not bounded, so they are not compact.

1.5 Cauchy Sequences

It is useful to have a criterion for convergence that does not explicitly involve the limit of the sequence.

Definition 14 *A sequence* $\{x_n\}$ *is a* Cauchy sequence *if* $\forall\, 1/m$, $\exists\, N$ *such that* $|x_i - x_j| < 1/m$ *whenever* $i, j \geq N$.

Theorem 6 *A sequence converges in the usual sense* $\iff$ *it is Cauchy.*

($\Rightarrow$) Assume $x_n \to x$: Given $1/m$, $\exists\, N$ such that if $n \geq N$, then $|x_n - x| < 1/(2m)$. Then $|x_i - x_j| \leq |x_i - x| + |x - x_j| < 1/(2m) + 1/(2m) = 1/m$ whenever $i, j \geq N$, so $\{x_n\}$ is Cauchy.

($\Leftarrow$) Assume $\{x_n\}$ is Cauchy. Given $1/m$, $\exists\, N$ such that $|x_i - x_j| < 1/m$ whenever $i, j \geq N$. Since

$$|x_i| = |x_i - x_N + x_N| \leq |x_N| + |x_i - x_N| < |x_N| + \frac{1}{m} \quad \forall\, i \geq N$$

we see that $\{x_n\}$ is bounded. Therefore, $s = \limsup\{x_n\}$ and $t = \liminf\{x_n\}$ are bounded. By definition of lim sup and lim inf, $\exists\, i, j \geq N$ such that $s - x_i < 1/m$ and $x_j - t < 1/m$. But then

$$0 \leq s - t = (s - x_i) + (x_j - t) + (x_i - x_j) \leq \frac{1}{m} + \frac{1}{m} + \frac{1}{m} = \frac{3}{m}$$

since m is arbitrary, we must have $s = t = \lim x_n$.

Example: Let $x_n = \log(n)$. Even though $|x_{n+1} - x_n| = \log(1 + 1/n) \to 0$, the sequence is *not* Cauchy, because $|x_i - x_j|$ must be small *for all* i, j large enough. In this case, $|x_i - x_j| = |\log(i/j)|$ can easily approach ∞ for large values of i.

1.6 Continuity

Definition 15 *A function f is* continuous *at x if it is defined in an interval around x and if $\forall\, 1/m$, $\exists\, 1/n$ (that may depend on $1/m$ and x) such that $|f(x) - f(y)| < 1/m$ whenever $|x - y| < 1/n$. We say f is continuous if is continuous at each point in its domain.*

By changing the condition $|x - y| < 1/n$ to $0 \le x - y < 1/n$ in the above definition we get the definition for *continuous from the left* at x. Similarly, changing $|x - y| < 1/n$ to $0 \le y - x < 1/n$ we get the definition for *continuous from the right* at x. We say that f is continuous on a closed interval $[a, b]$ if f is continuous on (a, b), continuous from the right at a, and continuous from the left at b.

f is continuous at x if and only if for all sequences $x_j \to x$ we have $f(x_j) \to f(x)$.

($\Rightarrow$) Given $1/m$, $\exists\, 1/n$ such that if $|x - y| < 1/n$ then $|f(x) - f(y)| < 1/m$. Also $\exists\, N$ such that if $j \ge N$ then $|x - x_j| < 1/n$. Putting these together gives $|f(x) - f(x_j)| < 1/m$ if $j \ge N$.

($\Leftarrow$) Suppose f is not continuous at x. Then, $\exists\, 1/m$ such that $\forall\, 1/n$ $\exists\, x_n$ satisfying $|x_n - x| < 1/n$ and $|f(x_n) - f(x)| \ge 1/m$. But $x_n \to x$, so by assumption, $|f(x_n) - f(x)| \to 0$, a contradiction. So, f must be continuous at x.

Definition 16 *A function f is* uniformly continuous *if $\forall\, 1/m$, $\exists\, 1/n$ (that depends only on $1/m$) such that $|f(x) - f(y)| < 1/m$ whenever $|x - y| < 1/n$.*

Theorem 7 *If f is continuous on $[a, b]$, then f is uniformly continuous.*

Suppose not. Then $\exists\, 1/m$ such that $\forall\, 1/n$, $\exists\, x_n, y_n$ satisfying $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \ge 1/m$. Since $[a, b]$ is compact, there exist convergent subsequences $x_{n'} \to x_0$ and $y_{n'} \to y_0$. But $|x_{n'} - y_{n'}| < 1/n'$, for an infinite number of $n' \to \infty$, so $x_0 = y_0$. Since f is continuous, $f(x_{n'}) \to f(x_0)$ and $f(y_{n'}) \to f(y_0)$, and hence $|f(x_{n'}) - f(y_{n'})| \to 0$, a contradiction!

Theorem 8 *f is continuous $\iff$ $f^{-1}(A)$ is open for all open sets A.*

($\Rightarrow$) Let $x \in f^{-1}(A)$. Since A is open, there is an interval contained in A around the point $f(x) \in A$. Thus, by choosing $1/m$ small enough, we may assume $(f(x) - 1/m, f(x) + 1/m) \subset A$. By the definition of continuity, $\exists\, 1/n$ such that $|f(x) - f(y)| < 1/m$ whenever $|x - y| < 1/n$. In other words, if $y \in (x - 1/n, x + 1/n)$ then $f(y) \in (f(x) - 1/m, f(x) + 1/m) \subset A$. This shows $(x - 1/n, x + 1/n) \subset f^{-1}(A)$ and hence that $f^{-1}(A)$ is open.

($\Leftarrow$) Fix x in the domain of f. For any $1/m$, the set $A = (f(x) - 1/m, f(x) + 1/m)$ is open, so by assumption, $f^{-1}(A)$ is open. Therefore, $f^{-1}(A)$ contains an interval around $x \in f^{-1}(A)$. Hence for $1/n$ small enough, $(x - 1/n, x + 1/n) \subset f^{-1}(A)$. This shows that if $|x - y| < 1/n$ (i.e.,

$y \in (x - 1/n, x + 1/n))$, then $|f(x) - f(y)| < 1/m$ (i.e., $f(y) \in A$), proving that f is continuous at x.

Theorem 9 *If f is continuous and A is compact, then $f(A)$ is compact.*

Let $f(A) \subset \bigcup U_\alpha$ be an open cover of $f(A)$. Then $\bigcup f^{-1}(U_\alpha)$ is an open cover of A by Theorem **??**. Since A is compact, there exists a finite subcover, $A \subset f^{-1}(U_{\alpha_1}) \cup \ldots \cup f^{-1}(U_{\alpha_n})$. But then $f(A) \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$ is a finite subcover of $f(A)$ proving that $f(A)$ is compact.

Theorem 10 (Extreme Value Theorem) *If f is continuous on $[a, b]$, then it has a maximum and a minimum on that interval.*

By Theorem **??**, $f([a, b])$ is compact, and hence closed and bounded by Theorem **??**. Therefore, $s = \sup\{f(x) \mid x \in [a, b]\}$ and $t = \inf\{f(x) \mid x \in [a, b]\}$ exist, and are finite limit points of $f([a, b])$. Consequently, there are sequences $\{x_n\}, \{y_n\} \subset [a, b]$ such that $f(x_n) \to s$ and $f(y_n) \to t$. Since $[a, b]$ is compact, there exists convergent subsequences $x_{n'} \to x_0$ and $y_{n'} \to y_0$. By Proposition **??**, $s = \lim f(x_n) = f(x_0)$ and $t = \lim f(y_n) = f(y_0)$. Therefore, $f(x_0)$ is a maximum of f and $f(y_0)$ is a minimum of f on $[a, b]$.

Theorem 11 (Intermediate Value Theorem) *Let f be continuous on $[a, b]$ and suppose y is a number between $f(a)$ and $f(b)$. Then $\exists, x \in [a, b]$ such that $f(x) = y$.*

We may assume without loss of generality that $f(a) < f(b)$. We repeatedly bisect the interval $[a, b]$ as follows. Let $c = (a + b)/2$. If $f(c) \leq y$, let $[a_1, b_1] = [c, b]$ and if $f(c) > y$, let $[a_1, b_1] = [a, c]$, so that $f(a_1) \leq y < f(b_1)$. Iterating this process we get a sequence of intervals $[a_n, b_n]$ satisfying $f(a_n) \leq y < f(b_n)$. Note that the length of these interval is shrinking geometrically:

$$b_n - a_n = (b_{n-1} - a_{n-1})/2 = \ldots = (b - a)/2^n$$

It follows that both sequences $\{a_n\}$ and $\{b_n\}$ are Cauchy and converge to a common limit $x \in [a, b]$. Therefore,

$$f(x) = \lim f(a_n) \leq y \leq \lim f(b_n) = f(x)$$

and $f(x) = y$ as claimed.

Note: The previous two theorems immediately imply that if f is continuous on $[a, b]$ then $f([a, b]) = [c, d]$ where c is the maximum of f and d is the minimum of f.