1 Uniform Convergence

1.1 Limits of Continuous Functions

In this section we consider a sequence of functions, $f_1(x),\ f_2(x),\ \ldots,$ that we assume share a common domain D. In order to understand the function obtained as the limit of this sequence, $f(x) = \lim_{n\to\infty} f_n(x)$ it would be useful to have criteria that help us decide such questions as

1. Is f continuous/integrable/differentiable if the f_n are?
2. Does $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$?
3. Does $f_n'(x) \to f'(x)$?

The concept of of *uniform convergence* plays a central role in such questions.

Definition 1 *A sequence of functions f_n* converges (pointwise) *to f, $f_n \to f$, if, for each $x \in D$, and $\forall\, 1/m$, $\exists\, N$ (that may depend on x and $1/m$) such that $|f_n(x) - f(x)| < 1/m$, $\forall\, n \geq N$*

Example: Let

$$f_n(x) = \begin{cases} 0 & 0 \leq x < 1 - \frac{1}{n} \\ \frac{n}{2}(x-1) + \frac{1}{2} & 1 - \frac{1}{n} \leq x < 1 + \frac{1}{n} \\ 1 & 1 + \frac{1}{n} \leq x \leq 2 \end{cases}$$

If $0 \leq x < 1/2$, then $\exists\, N$ such that $1 - 1/N < x$, so $f_n(x) = 0$, $\forall\, n \geq N$; if $1/2 < x \leq 1$, then $\exists\, N$ such that $1 + 1/N < x$, so $f_n(x) = 1$, $\forall\, n \geq N$; finally, if $x = 1/2$, then $f_n(x) = 1/2$, $\forall\, n$. Therefore, $f_n \to f$ where

$$f(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ \frac{1}{2} & x = 1 \\ 1 & \frac{1}{2} < x \leq 2 \end{cases}$$

Note that the limit function f is discontinuous, even though the functions f_n are continuous. A stronger version of convergence is needed to preserve continuity in the limit.

Definition 2 *A sequence of functions f_n* converges uniformly *to f, $f_n \to f$, if $\forall\, 1/m$, $\exists\, N$ (that may depend only on $1/m$) such that $|f_n(x) - f(x)| < 1/m$, $\forall\, n \geq N$ and $\forall\, x \in D$.*

Example: $f_n(x) = 1 + x + \ldots + x^n$ converges *uniformly* to $f(x) = \frac{1}{1-x}$ on $[-r, r]$ for any fixed $0 < r < 1$, not *not* on $(-1, 1)$.

$f_n(x) = (1 - x^{n+1})/(1-x) = f(x) - x^{n+1}/(1-x)$. If $|x| \leq r < 1$, then $1/|1-x| \leq 1/(1-r)$, so $|f_n(x) - f(x)| \leq |x|^{n+1}/(1-r) \leq r^{n+1}/(1-r)$. Now, given $1/m$, choose N such that $r^{N+1} < (1-r)/m$. Then, $\forall\, |x| \leq r$ and $\forall\, n \geq N$,

$$|f_n(x) - f(x)| \leq r^{n+1}/(1-r) \leq r^{N+1}/(1-r) < 1/m$$

To show the convergence is *not* uniform on $(-1,1)$, we must show that $\exists\, 1/m$ such that $\forall\, N$, $|f_n(x) - f(x)| \geq 1/m$ for some $x \in (-1,1)$ and some $n \geq N$. Since $|f_n(x) - f(x)| = |x|^{n+1}/|1-x| \rightarrow \infty$ as $x \rightarrow 1$, this is easy to arrange.

Theorem 1 *Assume f_n are continuous and $f_n \rightarrow f$ uniformly on D. Then f is continuous on D.*

Given $1/m$, choose N such that $|f_n(x) - f(x)| < 1/(3m)$, $\forall\, x \in D$ and $\forall\, n \geq N$. Then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{1}{3m} + |f_n(x) - f_n(y)| + \frac{1}{3m} \quad \forall\, x, y \in D \end{aligned}$$

Now fix $x \in D$. Since f_n is continuous at x, $\exists\, 1/k$ such that if $|x - y| < 1/k$ then $|f_n(x) - f_n(y)| < 1/(3m)$. Note that $1/k$ may depend on x. Then

$$|f(x) - f(y)| < \frac{1}{3m} + \frac{1}{3m} + \frac{1}{3m} = \frac{1}{m}$$

whenever $|x - y| < 1/k$ proving that f is continuous at x.

Note: The above proof also shows that if f_n are uniformly continuous and $f_n \rightarrow f$ uniformly, then f is uniformly continuous. The condition $|x - y| < 1/k$ in the proof can be removed and all inequalities hold $\forall\, x, y \in D$.

A useful modification of the definition of uniform convergence is *uniform convergence on compact subsets*. For example, $f_n(x) = 1 + x + \ldots + x^n$ converges uniformly on compact subsets of $(-1,1)$ as the previous example shows (any $[a,b] \subset (-1,1)$ is contained in $[-r,r]$ for some $r < 1$).

1.2 Limits of Integrals

The next theorem shows that we can interchange integration and *uniform* limits.

Theorem 2 *Assume f_n are integrable and $f_n \rightarrow f$ uniformly on $[a,b]$. Then f is integrable and $\lim_{n\rightarrow\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.*

Recall

$$\begin{aligned} \overline{I}(f) &= \inf\left\{ \int_a^b s(x)dx \;\middle|\; s(x) \geq f(x), s = \text{step function} \right\} \\ \underline{I}(f) &= \sup\left\{ \int_a^b t(x)dx \;\middle|\; t(x) \leq f(x), t = \text{step function} \right\} \end{aligned}$$

Given $1/m$, find N such that $|f_n(x) - f(x)| < 1/[3m(b-a)]$, $\forall\, x \in [a,b]$ and $\forall\, n \geq N$.

$$\begin{array}{rrcccl} \Rightarrow & f_n(x) - \frac{1}{3m(b-a)} & < & f(x) & < & f_n(x) + \frac{1}{3m(b-a)} \\ \Rightarrow & \underline{I}(f_n) - \frac{1}{3m} & \leq & \underline{I}(f) & \leq & \underline{I}(f_n) + \frac{1}{3m} \end{array}$$

and similarly for $\overline{I}(f)$ (by the linearity of $\underline{I}$ and $\overline{I}$). Therefore,

$$\begin{aligned} |\underline{I}(f) - \underline{I}(f_n)| &\leq \frac{1}{3m} \\ |\overline{I}(f) - \overline{I}(f_n)| &\leq \frac{1}{3m} \end{aligned}$$

Since $\overline{I}(f_n) = \underline{I}(f_n)$,

$$\begin{aligned} |\overline{I}(f) - \underline{I}(f)| &= |\overline{I}(f) - \overline{I}(f_n) + \underline{I}(f_n) - \underline{I}(f)| \\ &\leq |\overline{I}(f) - \overline{I}(f_n)| + |\underline{I}(f_n) - \underline{I}(f)| \\ &\leq \frac{1}{3m} + \frac{1}{3m} = \frac{2}{3m} < \frac{1}{m} \end{aligned}$$

Since $1/m$ is arbitrary, $\overline{I}(f) = \underline{I}(f) = I(f)$ so f is integrable and $I(f) = \int_a^b f(x)dx$. We have already shown $|I(f) - I(f_n)| \leq 1/(3m)$, $\forall\, n \geq N$, and this proves $I(f_n) \to I(f)$.

Example: Index the rational numbers in $[0,1]$ by $r_1, r_2, r_3, \ldots$ and define

$$f_n(x) = \begin{cases} 1 & \text{if } x = r_1, r_2, \ldots, r_n \\ 0 & \text{otherwise} \end{cases}$$

then $f_n(x) \to f(x)$ where

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \\ 0 & \text{otherwise} \end{cases}$$

Note that $\int_0^1 f_n(x)dx = 0$ and $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$, but $f(x)$ is not integrable!

1.3 Limits of Derivatives

The corresponding theorem for derivatives is *false*.

Example: Let $f_n(x) = \sqrt{x^2 + 1/n^2}$. Note that the functions $f_n(x)$ are differentiable and converge uniformly on $[-1,1]$ to $f(x) = \sqrt{x^2} = |x|$.

To see that the convergence is uniform, note that the maximum difference between f_n and f occurs at $x = 0$,

$$\max_{x\in[0,1]} |f_n(x) - f(x)| = \sqrt{0^2 + 1/n^2} - |0| = \frac{1}{n}$$

However, $f(x)$ is not differentiable at 0. To preserve differentiability in the limit we need to assume the uniform convergence of f_n'.

Theorem 3 *Assume* f_n *and* f_n' *are continuous on* $[a,b]$ *(i.e.,* $f_n \in C^1[a,b]$*). If* $f_n \to f$ *pointwise and* $f_n' \to g$ *uniformly, then* f *and* g *are continuous on* $[a,b]$ *and* $f' = g$*. (i.e.,* $f \in C^1[a,b]$*).*

Fix $x_0 \in [a,b]$. Then by the Fundamental Theorem of Calculus, $f_n(x) = f(x_0) + \int_{x_0}^{x} f_n'(t)dt$. By Theorem **??**, as $n \to \infty$, we get $f(x) = f(x_0) + \int_{x_0}^{x} g(t)dt$. Therefore $f'(x) = g(x)$.