1 Power Series

1.1 General Series

We first recall some facts about series from Calculus.

A *series*, $s = \sum_{n=0}^{\infty} u_n$, is defined as a limit, $s = \lim_{n\to\infty} s_n$, where s_n is the *n-th partial sum*, $s_n = u_0 + \cdots + u_n$. The series *converges* if the limit s exists and $|s| < \infty$; otherwise it *diverges*. We usually write simply $\sum u_n$ to denote a series with the starting limit implied by context and the upper limit ∞.

A series $\sum u_n$ converges *absolutely* if $\sum |u_n|$ converges. Recall that absolute convergence implies ordinary convergence. The terms of an absolutely convergent series may be rearranged in any manner without altering the sum. In contrast to this, the terms of a *conditionally convergent* series—convergent, but not absolutely convergent—may be rearranged to sum to any given value.

Convergence Tests

Simple Divergence Test If the terms u_n do not approach 0, then the series $\sum u_n$ diverges. (Equivalently, if $\sum u_n$ converges then $u_n \to 0$.)

Comparison Test Suppose $0 \le u_n \le w_n$, $\forall\, n \ge N$. Then $\sum w_n$ converges $\Rightarrow \sum u_n$ converges; $\sum u_n$ diverges $\Rightarrow \sum w_n$ diverges.

Root/Ratio Tests Assume $0 \le u_n$ and let $L = \lim_{n\to\infty} u_{n+1}/u_n$ or $L = \lim_{n\to\infty} \sqrt[n]{u_n}$. If $L < 1$ then the series $\sum u_n$ converges and if $L > 1$, the series diverges.

Geometric Series $\sum_{n=k}^{\infty} u^n$ converges to $u^k/(1-u)$ if $|u| < 1$ and diverges if $|u| \ge 1$.

The convergence of geometric series follows from the algebraic identity

$$1 + u + u^2 + \cdots + u^n = \frac{1}{1-u} - \frac{u^{n+1}}{1-u}$$

that gives an explicit formula for the n-th partial sum. The Root and Ratio Tests are derived from the geometric series and the Comparison Test.

1.2 Convergence of Power Series

Definition 1 *A* power series *is a series of the form*

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots$$

To keep the notation simple we shall assume without loss of generality that $x_0 = 0$.

If $\sum_{n=0}^{\infty} a_n x^n$ converges for one number, say b, then it converges absolutely and uniformly on $[-r, r]$ for any $0 \leq r < |b|$.

Since $\sum a_n b^n$ converges, the terms must approach 0, and hence $\exists\, M$ such that $|a_n b^n| \leq M$, $\forall\, n$. Then

$$|a_n x^n| = |a_n b^n||x^n/b^n| \leq M|x/b|^n$$

If $|x| \leq r < |b|$, then $|x/b| \leq r/|b| < 1$ and $\sum |a_n x^n|$ is dominated by a convergent geometric series, $\sum M(r/|b|)^n$. Note that the convergence depends only on $r/|b| < 1$ and not on any particular value of x. Therefore, $\sum a_n x^n$ converges absolutely and uniformly on $[-r, r]$.

This proposition shows if a power series converges, it must converge on a symmetric *interval* $(-R, R)$. The the largest such R is given a name.

Definition 2 *R is the* radius of convergence *of $\sum a_n x^n$ if the series converges for $|x| < R$ and diverges for $|x| > R$.*

In this definition we allow $R = 0$ (the series converges only for $x = 0$) or $R = \infty$ (the series converges for all x). The series may or may not converge at the endpoints $\pm R$. The value of R can often be conveniently found using the Root or Ratio Tests.

Examples: Let $\alpha > 0$.

1) Consider $\sum \alpha^n x^{2n}$.

We apply the Root Test to $\sum u_n$ where $u_n = (\alpha x^2)^n$. The limit $L = \lim_{n\to\infty} \sqrt[n]{u_n} = \alpha x^2$ and so, by the Root Test, the series converges if $|x| < 1/\sqrt{\alpha}$ and diverges if $|x| > 1/\sqrt{\alpha}$. Therefore, by definition, the radius of convergence is $R = 1/\sqrt{\alpha}$.

2) Consider $\sum \binom{\alpha}{n} x^n$ where $\alpha > 0$ is not an integer.

We apply the Ratio Test to $\sum u_n$ where $u_n = |\binom{\alpha}{n}||x|^n$. Since

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{|\alpha(\alpha-1)\cdots(\alpha-n)||x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|\alpha(\alpha-1)\cdots(\alpha-n+1)||x|^n} \\ &= \frac{|\alpha - n|}{n+1}|x| \longrightarrow |x| \end{aligned}$$

the Ratio Test implies the series converges if $|x| < 1$ and diverges if $|x| > 1$. Therefore the radius of convergence is $R = 1$.

3) Consider $\sum a_n x^n$ where

$$a_n = \begin{cases} 1 & n \text{ even} \\ \frac{1}{n^n} & n \text{ odd} \end{cases}$$

If we let $u_n = a_n|x|^n$, we find that neither u_{n+1}/u_n nor $\sqrt[n]{u_n}$ has a limit so the Root and Ratio Tests are not effective.

The next theorem shows a reliable way to obtain the radius of convergence R for any power series $\sum a_n x^n$.

Theorem 1 $1/R = \limsup |a_n|^{1/n}$

Step 1): Show $\limsup |a_n|^{1/n} \leq 1/R$.

The case $R = 0$ is trivial ($\limsup |a_n|^{1/n} \leq \infty$), so we assume $R > 0$. Choose $0 < r < R$. Since $\sum a_n r^n$ converges by the definition of R, $\exists\, M$ such that $|a_n r^n| \leq M$, $\forall\, n$. This implies

$$\limsup |a_n|^{1/n} \leq \frac{1}{r} \limsup M^{1/n} = \frac{1}{r}$$

Since this inequality is true $\forall\, r < R$, $\limsup |a_n|^{1/n} \leq 1/R$.

Step 2): Show $\limsup |a_n|^{1/n} \geq 1/R$.

Define R_0 by $1/R_0 = \limsup |a_n|^{1/n}$ (we allow $R_0 = 0$ or ∞). The case $R_0 = 0$ is trivial ($\infty \geq 1/R$), so we assume $R_0 > 0$. Fix r, $0 < r < R_0$, and choose R_1, $r < R_1 < R_0$. Since $1/R_0 < 1/R_1$, the definition of $\limsup$ implies $\exists\, N$ such that $|a_n|^{1/n} < 1/R_1$, $\forall\, n \geq N$. This implies

$$|a_n r^n| \leq \left|\frac{r}{R_1}\right|^n \quad \forall\, n \geq N$$

Since $|r/R_1| < 1$, $\sum |a_n r^n|$ is dominated by the convergent geometric series $\sum |r/R_1|^n$ and is itself convergent. Therefore, by the definition of radius of convergence, $r \leq R$. Consequently, $1/R \leq 1/r$, $\forall\, r < R_0$, and so $1/R \leq 1/R_0$.

Example: Consider $\sum a_n x^x$ where a_n is defined as in example 3) above. Since $\limsup |a_n|^{1/n} = 1$, the radius of convergence is $R = 1$.

Theorem 2 *Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R. Then $f(x)$ has derivatives of all orders. The derivatives are given by differentiating the series term by term,*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}$$

and have radius of convergence R. In particular,

$$a_n = \frac{f^{(n)}(0)}{n!}$$

By Proposition **??**, the polynomials $f_k(x) = \sum_{n=0}^{k} a_n x^n$ converge *uniformly* to $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| \leq r < R$ where R is the radius of convergence. Since $\limsup |n a_n|^{1/n} = \limsup |a_n|^{1/n}$, the radius of convergence of $g(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$ is still R by Theorem **??**. Thus the polynomials $f_k'(x) = \sum_{n=1}^{k} na_n x^{n-1}$ also converge uniformly to $g(x)$ for $|x| \le r < R$. By the Theorem **??** (convergence of derivatives), $f'(x) = g(x)$. We can repeat this argument to obtain derivatives of all orders. Finally, since $r < R$ was arbitrary, the above statements hold for $|x| < R$.

1.3 Expansions of Power Series

Theorem 3 *Suppose $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ has radius of convergence R. Then f has a power series expansion about any point x_1 in the interval $|x-x_0| < R$, and the radius of convergence of the new series is $\ge R - |x_1 - x_0|$.*

For simplicity of notation, we shall assume $x_0 = 0$. We use the Binomial Theorem to expand x^n in terms of $(x-x_1)^k$,

$$x^n = (x - x_1 + x_1)^n = \sum_{k=0}^{n} \binom{n}{k} x_1^{n-k}(x-x_1)^k$$

Then we plug this into the Taylor polynomial:

$$\begin{aligned} P_N(x) &= \sum_{n=0}^{N} a_n x^n \\ &= \sum_{n=0}^{N} a_n \sum_{k=0}^{n} \binom{n}{k} x_1^{n-k}(x-x_1)^k \\ &= \sum_{k=0}^{N} b_k (x-x_1)^k \quad \text{[rearranging terms]} \end{aligned}$$

where $b_k = \sum_{n=k}^{N} \binom{n}{k} a_n x_1^{n-k} = \sum_{n=0}^{N-k} \binom{n+k}{k} a_{n+k} x_1^n$. The following diagram helps to see how the indexes have been rearranged. The indexing of the original sum follows the vertical lines from $k=0$ to $k=n$ with n running from 0 to N. The rearrangement indexes the same shaded region horizontally from $n=k$ to $n=N$ with k running from 0 to N.

This suggests that $f(x) = \sum_{k=0}^{\infty} b_k(x-x_1)^k$ where

$$b_k = \sum_{n=0}^{\infty} \binom{n+k}{k} a_{n+k} x_1^n$$

It is interesting to note that this formula for b_k could be obtained by computing derivatives:

$$f^{(k)}(x_1) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n x_1^{n-k}$$

$$\frac{f^{(k)}(x_1)}{k!} = \sum_{n=k}^{\infty} \binom{n}{k} a_n x_1^{n-k} = \sum_{n=0}^{\infty} \binom{n+k}{k} a_{n+k} x_1^n = b_k$$

Note that the above series for b_k converges for $|x_1| < R$ since the factor $\binom{n+k}{k} = (n+k)(n+k-1)\cdots(n+1)/k!$ is a polynomial in n and therefore $\binom{n+k}{k}^{1/n} \to 1$ as $n \to \infty$ (apply Theorem **??**).

To prove that the series $\sum_{k=0}^{\infty} b_k(x-x_1)^k$ converges to $f(x)$ we first show that the following series, indexed by the *pairs* (n,k), $k \le n$, converges absolutely:

$$s(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} a_n x_1^{n-k}(x-x_1)^k$$

Let $r = |x-x_1| + |x_1|$. Then

$$\sum_{k=0}^{n} \binom{n}{k} |x-x_1|^{n-k}|x_1|^k = (|x-x_1| + |x_1|)^n = r^n$$

Therefore,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} |a_n||x_1|^{n-k}|x-x_1|^k = \sum_{n=0}^{\infty} |a_n| r^n$$

converges for $r < R$ (equivalently for $|x-x_1| < R-|x_1|$). Since the series $s(x)$ converges absolutely, we may rearrange the order of summation. Summing k first and then n gives

$$s(x) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} x_1^{n-k}(x-x_1)^k = f(x)$$

Summing n first and then k gives

$$s(x) = \sum_{k=0}^{\infty}\Big(\sum_{n=k}^{n} \binom{n}{k} x_1^{n-k}\Big)(x-x_1)^k = \sum_{k=0}^{\infty} b_k(x-x_1)^k$$

Therefore, $f(x) = \sum_{k=0}^{\infty} b_k(x-x_1)^k$ for $|x-x_1| < R-|x_1|$.

Definition 3 *A function is* analytic *if it has a power series expansion about every point in its domain.*

1.3.1 Facts from Calculus

- Most important functions in mathematics are analytic, e.g.,

- polynomials
- trigonometric functions
- exponential and logarithm functions

- If f and g are analytic, so are $f \pm g$, $f \cdot g$, f/g (wherever $g(x) \neq 0$) and $f \circ g$ are analytic.
- *Taylor's Theorem*: If f is any function with derivatives up to order $n+1$ at x_0, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad + \quad R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

for some c between x_0 and x, $|c-x_0| < |x-x_0|$. In particular, f is analytic at x_0 if and only if $|R_n(x)| \to 0$ as $n \to \infty$ for x in some interval around x_0.

1.3.2 A Fact from Complex Analysis.

If f is analytic, then the radius of convergence for the power series expansion around x_0 is the distance of x_0 to the "nearest singularity" in the complex plane . A series with complex numbers converges for all complex numbers within a circle of radius R of the center, and that is the origin of the term *radius* of convergence.

Example: $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$. The radius of convergence is $R = 1$ even though $\frac{1}{1+x^2}$ is defined for all $x \in$. But in , $\frac{1}{1+x^2}$ has singularities at $x = \pm i$ and the distance from 0 to $\pm i$ is $1 = R$.

1.4 Analytic Continuation

A remarkable fact about an analytic function f is that its values on an interval (a, b) are determined by its values on any neighborhood contained in (a, b), no matter how small. We are *not* assuming that f has one power series expansion whose interval of convergence covers the entire interval (a, b), so we cannot simply quote Theorem **??** to prove this statement.

What we do assume is that f has a power series expansion at each point of (a, b) with varying, and perhaps very small, radii of convergence. The process of linking these series together from one interval of convergence to the next is called *analytic continuation*. We shall now sketch how to do this on (a, b).

Let (c, d) be an interval contained in the given neighborhood where we know the values of $f(x)$. Let $p \in (a, b) \setminus (c, d)$. We must find the value of $f(p)$ using the values of $f(x)$ for $x \in (c, d)$.

Let us assume $p > d$ (a similar argument would handle the case $p < c$). Let R_1 be the radius of convergence of the power series expansion of f about $x_1 = d$. Choose $x_0 \in (x_1 - R_1, x_1)$ close to $x_1 = d$. By Theorem **??**, the radius of convergence of the power series expansion of f about x_0 is $R_0 \geq R_1 - (x_1 - x_0)$. In particular, $x_1 \in (x_0 - R_0, x_0 + R_0)$. Therefore, the known values of $f^{(n)}(x_0)$ determine the values of $f^{(n)}(x_1)$, and these in turn determine the values of f (through the power series expansion at x_1) in $(c, d) \cup (x_1 - R_1, x_1 + R_1)$.

Thus, we can always extend the interval of the known values of f past the endpoint of an interval of previously known values (as long as we remain in (a, b)). Repeating this process, we obtain a sequence of centers, $x_i = x_{i-1} + R_{i-1}$ with radii of convergence R_i. We claim that $p \in (x_i - R_i, x_i + R_i)$ for some i. Otherwise, the centers must converge $x_i \to q$ and the radii approach 0, $R_i \to 0$. But f has a power series expansion at q with radius of convergence Q and the intervals $(x_i - R_i, x_i + R_i)$ must eventually fall inside $(q - Q, q + Q)$. For example, there is an x_i with $|x_i - q| < Q/2$ and $R_i < Q/2$. This contradicts Theorem **??** which asserts that the radius of R_i should be at least $Q - |x_i - q| > Q/2$.