1 Approximation by Polynomials

We know that if f is analytic then it can be approximated by Taylor polynomials, $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$. In fact $P_n(x) \to f(x)$ uniformly on compact subsets of its interval of convergence.

Can continuous functions be approximated by polynomials?

1.1 Lagrange Interpolation

Let f be continuous on $[a, b]$ and pick $x_0, \ldots, x_n \in [a, b]$. Then there is a polynomial P_n of degree n such that $P_n(x_j) = f(x_j)$ for $j = 0, \ldots, n$.

Here is how to construct P_n: Define $q_k(x) = \Pi_{j \neq k}(x - x_j)$. Then $q_k(x_j) = 0$ if $j \neq k$ and $q_k(x_k) \neq 0$. Next define

$$Q_k(x) = \frac{q_k(x)}{q_k(x_k)} \quad \text{so} \quad Q_k(x_j) = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

Finally, letting, $a_k = f(x_k)$, we see that the polynomial

$$P_n(x) = \sum_{k=0}^{n} a_k Q_k(x)$$

has the desired properties.

Example: If $f(x) = \sin(4x)$ and $x_k = k$, for $k = 0, \ldots, 5$, the 5th degree polynomial obtained by Lagrange interpolation does not approximate $f(x)$ well.

We could hope that $P_n \to f$ as $n \to \infty$, although, it is not clear how to prove this. A better idea is to use *convolution with polynomials*.

1.2 Convolution

Definition 1 *The* support *of a function f is the closure of the set of points x in the domain of f such that $f(x) \neq 0$.*

Note: If the support of a function f is compact then f must vanish outside a bounded interval: $f(x) = 0$, $\forall\, x \notin [a, b]$.

Definition 2 *Let f and g be integrable functions and assume either f or g has compact support. The* convolution *of f and g is*

$$f * g(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

Note: By making a change of variable, $u = x - t$, $t = x - u$, is it easy to show that $f * g = g * f$.

Theorem 1 *Let be f integrable and have compact support. If g is a polynomial of degree n, then $f * g$ is a polynomial of degree $\leq n$.*

Let $g(x) = \sum_{k=0}^{n} a_k x^k$ and suppose $f(x) = 0$, $\forall\, x \notin [a, b]$. Then

$$\begin{aligned}
f * g(x) &= g * f(x) = \int_a^b g(x-t)f(t)\,dt \\
&= \int_a^b \sum_{k=0}^{n} a_k (x-t)^k f(t)\,dt \\
&= \int_a^b \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} a_k \binom{k}{j} x^j t^{k-j} f(t)\,dt \\
&= \sum_{j=0}^{n} b_j x^j
\end{aligned}$$

where

$$b_j = \sum_{k=j}^{n} \left\{ (-1)^{k-j} a_k \binom{k}{j} \int_a^b t^{k-j} f(t)\,dt \right\} \in$$

Theorem 2 *Let f be C^1 and have compact support. Let g be continuous. Then $(f * g)' = f' * g$.*

We compute the derivative at x_0 using an arbitrary sequence $x_n \to x_0$:

$$\begin{aligned}
(f * g)'(x_0) &= \lim_{n\to\infty} \frac{f * g(x_n) - f * g(x_0)}{x_n - x_0} \\
&= \lim_{n\to\infty} \frac{1}{x_n - x_0} \left[\int f(x_n - t)g(t)\,dt - \int f(x_0 - t)g(t)\,dt \right] \\
&= \lim_{n\to\infty} \int \frac{f(x_n - t) - f(x_0 - t)}{x_n - x_0} g(t)\,dt \\
&= \lim_{n\to\infty} \int f'(z_n - t)g(t)\,dt
\end{aligned}$$

for some z_n between x_0 and x_n by the Mean Value Theorem. We now show that $f'(z_n - t)g(t) \to f(x_0 - t)g(t)$ *uniformly* so that we may interchange the limit and the integral, by Theorem **??**, to get

$$\begin{aligned}(f*g)'(x_0) &= \int \lim_{n\to\infty} f'(z_n-t)g(t)\,dt = \int f'(x_0-t)g(t)\,dt \\ &= (f'*g)(x_0)\end{aligned}$$

Since f has compact support, say contained in $[a,b]$, the support of f' must necessarily fall in that same interval and therefore is compact (if $f\equiv 0$ then $f'\equiv 0$). By Theorem **??**, f' is uniformly continuous on $[a,b]$, and hence uniformly continuous on since $f'\equiv 0$ outside $[a,b]$. Since $z_n \to x_0$, there is some closed interval, $[c,d]$, that contains all z_n and x_0. Then $z_n - t, x_0 - t \in [a,b] \Rightarrow t \in [c-b, d-a]$. Let M be the maximum of g on $[c-b,d-a]$.

Since f' is uniformly continuous, $\forall\, 1/m$, $\exists\, 1/k$ such that

$$|f'(z_n-t)-f'(x_0-t)| < \frac{1}{m\cdot M}$$

whenever

$$|(z_n-t)-(x_0-t)| = |z_n-x_0| < \frac{1}{k}$$

However, $\exists\, N$ such that

$$|z_n-x_0| < \frac{1}{k} \quad \forall\, n\ge N$$

Therefore, if $n\ge N$, then

$$|f'(z_n-t)g(t)-f'(x_0-t)g(t)| \le |f'(z_n-t)-f'(x_0-t)|M < \frac{1}{m} \quad \forall\, t$$

and hence $f'(z_n-t)g(t)\to f(x_0-t)g(t)$ uniformly.

1.3 Weierstrass Approximation Theorem

Theorem 3 *Let f be continuous on $[a,b]$. Then there exists a sequence of polynomials converging uniformly to f on $[a,b]$.*

We may assume $[a,b]=[0,1]$, for if the theorem is true for

$$g(t) = f(a+(b-a)t) \quad t\in[0,1]$$

then there exists polynomials $Q_m(t)\to g(t)$ uniformly on $[0,1]$ and this implies that the polynomials $P_m(x) = Q_m((x-a)/(b-a)) \to f(x)$ uniformly on $[a,b]$. We may also assume $f(0)=f(1)=0$, for if the theorem is true for

$$g(x) = f(x) - f(0) - x(f(1)-f(0))$$

then there exist polynomials $Q_m \to g$ uniformly on $[0,1]$ and this implies that the polynomials $P_m(x) = Q_m(x) + f(0) - x(f(1)-f(0)) \to f(x)$ uniformly on $[0,1]$. We extend $f(x)$ to a continuous function on by $f(x)=0$, $\forall\, x\notin[0,1]$.

Define $h_m(x) = c_m^{-1}(1-x^2)^m$ where $c_m = \int_{-1}^1 (1-x^2)^m dx$, so

$$\int_{-1}^1 h_m(x)\,dx = 1 \tag{1}$$

In order to estimate h_m we need a lower bound for c_m:

$$c_m = \int_{-1}^1 (1-x^2)^m dx = 2\int_0^1 (1-x^2)^m dx \geq 2\int_0^{1/\sqrt{m}} (1-x^2)^m dx$$

It is easy to verify the inequality $1 - mx^2 \leq (1-x^2)^m$ for $x \in [0,1]$: If $q(x) = (1-x^2)^m - (1-mx^2)$ then $q(x) \geq 0$ because $q(0) = 0$ and $q'(x) \geq 0$ for $x \in [0,1]$. Inserting this inequality above gives

$$c_m \geq 2\int_0^{1/\sqrt{m}} (1-mx^2)dx = \frac{4}{3\sqrt{m}} > \frac{1}{\sqrt{m}}$$

Since, $c_m^{-1} < \sqrt{m}$ we see that $h_m \to 0$ uniformly for $1/n \leq |x| \leq 1$:

$$h_m(x) = c_m^{-1}(1-x^2)^m \leq \sqrt{m}\Big(1 - \frac{1}{n^2}\Big)^m \to 0, \quad m \to \infty \tag{2}$$

On the other hand $\int_{-1}^1 h_m(x)\,dx = 1$, so the graph of $h_m(x)$ is more and more concentrated at 0 as $m \to \infty$. In fact, we may think of the limit as the Dirac delta function, $\lim_{m\to\infty} h_m(x) = \delta_0(x)$.

Define $P_m(x) = \int_{-1}^1 f(x-1)h_m(t)\,dt = f * h_m(x)$ for $x \in [0,1]$. By Theorem **??**, P_m is a polynomial of degree $\leq 2m$. We now show that $P_m \to f$ uniformly.

Given $\epsilon > 0$, choose $1/n$ such that if $|y-x| < 1/n$, then $|f(y) - f(x)| < \epsilon/2$ (uniform continuity). Using the fact that $\int_{-1}^1 h_m(t)\,dt = 1$, we get

$$\begin{aligned} |P_m(x) - f(x)| &= \Big|\int_{-1}^1 f(x-t)h_m(t)\,dt - f(x)\int_{-1}^1 h_m(t)\,dt\Big| \\ &= \Big|\int_{-1}^1 (f(x-t) - f(x))h_m(t)\,dt\Big| \\ &\leq \int_{-1}^1 |f(x-t) - f(x)|h_m(t)\,dt \end{aligned}$$

Now, $|f(x-t) - f(t)| < 2M$ where M is the maximum of f on $[0,1]$ and $|f(x-t) - f(x)| < \epsilon/2$ if $|(x-t) - x| = |t| < 1/n$. So, breaking $[-1,1]$ up into $[-1,-1/n] \cup [-1/n, 1/n] \cup [1/n, 1]$, we get

$$|P_m(x) - f(x)| \leq 2M\int_{-1}^{-1/n} h_m(t)\,dt + \frac{\epsilon}{2}\int_{-1/n}^{1/n} h_m(t)\,dt + 2M\int_{1/n}^1 h_m(t)\,dt$$

The first and third integrals are $\leq 2M\sqrt{m}(1-1/n^2)^m(1-1/n) < 2M\sqrt{m}(1-1/n^2)^m$ by estimate (**??**). The middle integral is $< \epsilon/2$ by property (**??**). So

$$|P_m(x) - f(x)| \leq 4M\sqrt{m}\Big(1 - \frac{1}{n^2}\Big)^m + \frac{\epsilon}{2} < \epsilon$$

for m large enough and for all $x \in [0,1]$. Therefore, $P_m \to f$ uniformly on $[0,1]$.

One of the advantages of using convolution for polynomial approximations is that it gives more information about derivatives.

Corollary 4 *If $f \in C^1[a,b]$, then there exists a sequence of polynomials P_m such that $P_m \to f$ and $P'_m \to f'$ uniformly on $[a,b]$.*

As in the previous proof, we may assume that $[a,b] = [0,1]$. Furthermore, by subtracting the cubic polynomial

$$[2f(0) - 2f(1) + f'(0) + f'(1)]x^3$$
$$+ [3f(1) - 3f(0) + 2f'(0) - 2f'(0) - f'(1)]x^2 + f'(0)x + f(0)$$

from f we may assume f and f' can be extended continuously by 0 to . Let $P_m = f * h_m$ as above. By Theorem **??**, $P'_m = f' * h_m$. Therefore both $P_m \to f$ and $P'_m \to f'$ uniformly on $[a,b]$ by the previous proof.