1 Equicontinuity

1.1 Existence of Convergent Subsequences

The following type of problem frequently arises in analysis: Given a family of functions, $\mathcal{F} \subset C[0,1]$, can we find a function $f_0 \in C[0,1]$ ($f_0 \in \mathcal{F}$?) that minimizes some quantity $E(f)$ over all $f \in \mathcal{F}$.

For example, let $E(f) = \int_0^1 |f(x)|^2 dx$. Since $E(f) \geq 0$, $\forall\, f \in \mathcal{F}$, $m = \inf_{f \in \mathcal{F}} E(f)$ exists and so $\exists\, \{f_n\} \subset \mathcal{F}$ such that $E(f_n) \to m$. Now if we could find a subsequence $\{f_{n'}\}$ that converged *uniformly* to f_0, then f_0 would be continuous by Theorem **??** and $E(f_{n'}) \to E(f_0) = m$.

Not all sequences $\{f_n\} \subset C[a,b]$ have convergent subsequences.

Example: $f_n(x) = n$.

This problem does not occur if we can put a uniform bound on every function in the sequence.

Definition 1 *A family of functions, $\mathcal{F}$, is* uniformly bounded *if $\exists\, M$ such that $|f(x)| \leq M$, $\forall\, x \in [a,b]$ and $\forall\, f \in \mathcal{F}$.*

Not all *bounded* sequences $\{f_n\} \subset C[a,b]$ have convergent subsequences.

Example: $f_n(x) = \sin(nx)$.

This problem does not occur if the oscillations (or the "continuity") of the functions in the sequence can be controlled in a uniform way.

Definition 2 *A family of functions, $\mathcal{F}$, is* uniformly equicontinuous *if $\forall\, 1/m$, $\exists\, 1/n$ (depending only on $1/m$) such that*

$$|x - y| < \frac{1}{n} \Rightarrow |f(x) - f(y)| < \frac{1}{m}, \quad \forall\, f \in \mathcal{F}$$

1.2 Arzela-Ascoli Theorem

Theorem 1 *If $\{f_n\} \subset C[a,b]$ is uniformly bounded and uniformly equicontinuous, then there is a subsequence $f_{n'}$ converging uniformly to $f \in C[a,b]$.*

Let $x_1, x_2, x_3, \ldots$ be a countable dense subset of $[a,b]$, e.g., $[a,b] \cap$. Then for each fixed k, $f_1(x_k), f_2(x_k), f_3(x_k), \ldots$ is a bounded sequence ($|f_n(x)| \leq M, \forall\, x \in [a,b], \forall\, n$). So, for each k, there is a convergent subsequence, $f_{n'}(x_k) \to c_k$.

From such subsequences we want to extract a subsequence that converges simultaneously $\forall\, k$. We do this as follows:

$$\begin{array}{lcl} \{f_{1i}\} \subset \{f_n\} & : & \boxed{f_{11}(x_1)}, f_{12}(x_1), f_{13}(x_1), \ldots \to c_1 \\ \{f_{2i}\} \subset \{f_{1i}\} & : & f_{21}(x_2), \boxed{f_{22}(x_2)}, f_{23}(x_2), \ldots \to c_2 \\ \{f_{3i}\} \subset \{f_{2i}\} & : & f_{31}(x_3), f_{32}(x_3), \boxed{f_{33}(x_3)}, \ldots \to c_3 \end{array}$$

We take the diagonal subsequence of functions, $f_{11},\ f_{22},\ f_{33},\ \ldots$. Except for the first k functions, $\{f_{ii}\}$ is a subsequence of the kth row, $\{f_{ki}\}$. Therefore, $\{f_{ii}\}$ converges simultaneously on all x_n.

We now show $\{f_{ii}\}$ converges uniformly on $[a,b]$ (the limit function is automatically continuous by Theorem **??**). We will prove the *uniform Cauchy criterion* holds:

Given $1/m$, $\exists\, N$ such that $|f_{kk}(x) - f_{jj}(x)| < 1/m$, $\forall\, x \in [a,b]$ and $\forall\, j,k \geq N$.

The assumption of uniform equicontinuity implies $\exists\, 1/n$ such that

$$|x-y| < \frac{1}{n} \Rightarrow |f_j(x) - f_j(y)| < \frac{1}{3m} \quad \forall\, j \tag{1}$$

Since $\bigcup(x_k - 1/n, x_k + 1/n)$ covers the compact interval $[a,b]$, there is a finite subcover. To keep the notation simple, we re-index the x_k's so that

$$[a,b] \subset (x_1 - \frac{1}{n_1}, x_1 + \frac{1}{n_1}) \cup \ldots \cup (x_r - \frac{1}{n_r}, x_r + \frac{1}{n_r})$$

We know $f_{ii}(x_p)$ converges for $p = 1, \ldots, r$. So $\exists\, N$ such that

$$|f_{kk}(x_p) - f_{jj}(x_p)| < \frac{1}{3m}, \forall\, j,k \geq N, p = 1, \ldots, r \tag{2}$$

Conclusion: If $j,k \geq N$ and $x \in [a,b]$, then $\exists\, x_p$ such that $|x - x_p| < 1/n$. Therefore, applying (**??**) and (**??**),

$$\begin{aligned} & |f_{kk}(x) - f_{jj}(x)| \\ & \quad \leq \quad |f_{kk}(x) - f_{kk}(x_p)| + |f_{kk}(x_p) - f_{jj}(x_p)| + |f_{jj}(x_p) - f_{jj}(x)| \\ & \quad < \quad \frac{1}{3m} + \frac{1}{3m} + \frac{1}{3m} = \frac{1}{m} \end{aligned}$$

1.3 Criterion for Equicontinuity

Theorem 2 *If* $\{f_k\} \subset C^1[a,b]$ *and* $|f'_k(x)| \leq M$, $\forall\, k$, *and* $x \in [a,b]$, *then* $\{f_k\}$ *is uniformly equicontinuous.*

Given $1/m$, let $n = m \cdot M$. Then $\forall\, x, y \in [a, b]$, the Mean Value Theorem implies there is a z between x and y such that

$$f_k'(z) = \frac{f_k(x) - f_k(y)}{x - y}$$

Therefore,

$$|f_k(x) - f_k(y)| = |f_k'(z)||x - y| \leq M|x - y| < \frac{1}{m}$$

whenever $|x - y| < 1/n$.

Corollary 3 *Suppose* $\{f_k\} \subset C^1[a, b]$, *and* $\exists\, M$ *such that* $|f_k(x)| \leq M$ *and* $|f_k'(x)| \leq M$, $\forall\, k$ *and* $x \in [a, b]$. *Then there is a subsequence* $\{f_{k'}\}$ *converging uniformly* $f_{k'} \to f \in C[a, b]$.

Remember: uniform means "for all x"; *equi-* means "for all k"