1 Structures on Euclidean Space

Euclidean space is the set of ordered n-tuples of real numbers,

$$^n = \{(x_1, \ldots, x_n) \mid x_i \in\}$$

In this section we shall examine various "structures" on ^{n}:

- vector space
- metric space
- normed space (Banach space)
- inner product space (Hilbert space)

1.1 Vector Spaces

Definition 1 *A* vector space *over a field is a set V equipped with two operations:*

Vector Addition $\forall\, v, w \in V$, $\exists\, v + w \in V$*; commutative, associative, identity element (0), and inverses exist ($v + (-v) = 0$).*

Scalar Multiplication $\forall\, a \in$, $v \in V$, $\exists\, a \cdot v \in V$*; associative and distributive.*

Example: Euclidean space becomes a vector space by defining addition and scalar multiplication by

$$\begin{aligned} x + y &= (x_1 + y_1, \ldots, x_n + y_n) \\ ax &= (ax_1, \ldots, ax_n) \end{aligned}$$

for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in^n$ and $a \in$.

A standard *basis* for ^{n} is $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, \ldots, 0, 1)$. Thus $x \in^n$ can be written $x = x_1 e_1 + \ldots + x_n e_n$. The dimension of ^{n} is $\dim^n = n$, but a vector space can have infinite dimension.

Example: $V = C[a, b]$ with normal addition of functions and scalar multiplication is an infinite dimensional vector space over .

1.2 Metric Spaces

Definition 2 *A* metric space *is a set M equipped with a distance function $d : M \times M \rightarrow$ satisfying*

1. $d(x, y) \geq 0$ *with "=" iff* $x = y$ *(*positivity*)*
2. $d(x, y) = d(y, x)$ *(*symmetry*)*
3. $d(x, z) \leq d(x, y) + d(y, z)$ *(*triangle inequality*)*

$\forall\, x, y, z \in M.$

Example: ^{n} with Euclidean distance

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Properties 1 and 2 are obvious. The proof of property 3 will be a consequence of a general theorem proved later.

Example: $C[a, b]$ with distance

$$d(f, g) = \int_a^b |f(x) - g(x)|\, dx$$

All three properties are easy to verify.

1.3 Normed Spaces

Definition 3 *A* normed space *(or* Banach space*) is a vector space* V *over equipped with a norm* $\|\cdot\| : v \rightarrow$ *satisfying:*

1. $\|x\| \geq 0$ *with "=" iff* $x = 0$ *(*positivity*)*
2. $\|ax\| = |a| \cdot \|x\|$ *(*homogeneity*)*
3. $\|x + y\| \leq \|x\| + \|y\|$ *(*triangle inequality*)*

$\forall\, x, y \in V,\ a \in.$

Example: ^{n} is a normed space with the Euclidean norm

$$|x| = \sqrt{x_1^2 + \ldots + x_n^2}$$

This norm is related to the Euclidean distance by $d(x, y) = |x - y|$.

A norm measures *length*.

Remark: If V is a normed space, then it has a metric structure induced by the norm

$$d(x, y) = \|x - y\|$$

- positivity: $d(x,y) = \|x-y\| \geq 0$
- homogeneity $\Rightarrow$ symmetry:

$$\begin{aligned} d(x,y) &= \|x-y\| = \|(-1)(y-x)\| \\ &= |-1| \cdot \|y-x\| \\ &= \|y-x\| = d(y,x) \end{aligned}$$

- triangle inequality:

$$\begin{aligned} d(x,z) &= \|x-z\| = \|(x-y)+(y-z)\| \\ &\leq \|x-y\| + \|y-z\| \\ &= d(x,y) + d(y,z) \end{aligned}$$

1.3.1 Other norms on ^{n}

- $\|x\|_1 = \sum_{j=1}^n |x_i|$ ($\|x-y\|_1$ is the "taxi-cab distance" between x and y)
- $\|x\|_p = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}$, $1 \leq p < \infty$
- $\|x\|_{\sup} = \max_j\{|x_j|\}$

It is non-trivial to prove the triangle inequality for $\|x\|_p$ when $p \neq 1, 2$. The proof for $p = 2$ follows from the Cauchy-Schwartz Inequality below. The sup-norm can be considered the case "$p = \infty$":

$$\left(\sum_{j=1}^n |x_j|^p\right)^{1/p} = |x_k|\left(\frac{|x_1|^p}{|x_k|^p} + \cdots + 1 + \cdots + \frac{|x_n|^p}{|x_k|^p}\right)^{1/p}$$

If $|x_k| = \max_j |x_j| = \|x\|_{\sup}$, then $|x_j|^p/|x_k|^p \to 0$ or 1, so

$$|x_k|\left(\frac{|x_1|^p}{|x_k|^p} + \cdots + 1 + \cdots + \frac{|x_n|^p}{|x_k|^p}\right)^{1/p} \to |x_k|(1 + \cdots + 1)^0 = \|x\|_{\sup}$$

In fact, the sup-norm is sometimes denoted $\|x\|_\infty$.

$$\begin{aligned} \|x\|_\infty &= 1 \quad \text{square} \\ \|x\|_2 &= 1 \quad \text{circle} \\ \|x\|_1 &= 1 \quad \text{diamond} \end{aligned}$$

Example: $C[a,b]$ is a vector space over (dim $= \infty$). It is also a normed space with either

- $\|f\|_{\sup} = \sup\{|f(x)| \mid x \in [a,b]\}$

- $\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}, \; 1 \le p < \infty$

The same remarks apply to $\|f\|_p$ as apply to $\|x\|_p$ above.

1.4 Inner Product Spaces

Definition 4 *An* inner product space *(or* Hilbert space*) is a vector space V over with a function $\cdot, \cdot : V \times V \to$ satisfying*

1. $x, x \ge 0$ *with "=" iff* $x = 0$ *(*positive definite*)*
2. $x, y = y, x$ *(*symmetry*)*
3. $ax + by, z = ax, y + by, z$ *(*bilinear*)*

$\forall\, a, b \in, \; x, y, z \in V$.

Example: An inner product on ^{n} can be defined by

$$x, y = x_1 y_1 + \cdots x_n y_n$$

This inner product is often written $x \cdot y$ and called the *dot product.* It is related to the angle θ between x and y by the formula

$$x \cdot y = |x|\,|y|\,\cos(\theta)$$

We can demonstrate this for $n = 2$ using the addition formulas for sine and cosine. If $x = (r\cos(\alpha), r\sin(\alpha))$ and $y = (R\cos(\beta), R\sin(\beta))$, where $r = |x|$ and $R = |y|$, then

$$\begin{aligned} x \cdot y &= rR(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)) \\ &= rR\cos(\beta - \alpha) \\ &= |x|\,|y|\,\cos(\theta) \end{aligned}$$

In ^{n} it is clear that the dot product is related to the Euclidean norm: $|x| = \sqrt{x \cdot x}$. This is a special case of a general fact: an inner product $\cdot, \cdot$ always defines a norm by $\|x\| = \sqrt{x, x}$. To prove this we need the following important theorem.

Theorem 1 (Cauchy-Schwartz Inequality) *If V is a vector space over with inner product $\cdot, \cdot$, then*

$$|x, y| \le \sqrt{x, x}\,\sqrt{y, y}$$

with "=" iff x and y are collinear (i.e., $x = cy$ or $y = cx$ for some $c \in$).

$x+y, x+y \ \geq 0$ and $x-y, x-y \ \geq 0$ so using bilinearity,

$$\begin{array}{rcrl} x,x \ + 2x,y \ + y,y \ \geq 0 & \Rightarrow & -x,y & \leq \frac{1}{2}(x,x \ + y,y \ ) \\ x,x \ - 2x,y \ + y,y \ \geq 0 & \Rightarrow & x,y & \leq \frac{1}{2}(x,x \ + y,y \ ) \end{array}$$

Therefore,

$$|x,y \ | \leq \frac{1}{2}(x,x \ + y,y \ ) \tag{1}$$

Now write

$$\begin{array}{lll} x = a \cdot u & \text{where} & a = \sqrt{x,x \ }, \quad u = a^{-1} \cdot x \\ y = b \cdot v & \text{where} & b = \sqrt{y,y \ }, \quad v = b^{-1} \cdot y \end{array}$$

(We may assume $x \neq 0$ and $y \neq 0$ for otherwise the theorem is trivial.)

Note that $u,u \ = a^{-2}x,x \ = 1$ and $v,v \ = b^{-2}y,y \ = 1$. Using bilinearity and applying (**??**) to u and v, we get

$$\begin{array}{rcl} |x,y \ | & = & |au,bv \ | = ab|u,v \ | \\ & \leq & ab\frac{1}{2}(u,u \ + v,v \ ) = ab\frac{1}{2}(1+1) = ab \\ & = & \sqrt{x,x} \sqrt{y,y} \end{array}$$

To prove the last assertion of the theorem, assume "=" holds. Then $|u,v \ | = 1$, where u and v are as above. If $u,v \ = 1$, then $u-v, u-v \ = u,u \ - 2u,v \ + v,v \ = 0$, while if $u,v \ = -1$, then $u+v, u+v \ = u,u \ + 2u,v \ + v,v \ = 0$. In either case we get $u = \pm v$ and hence $x = \pm ab^{-1}y$.

Corollary 2

$$\Big|\sum_{i=1}^{n} x_i y_i\Big| \leq \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}\Big(\sum_{i=1}^{n} y_i^2\Big)^{1/2}$$

Corollary 3

$$\Big|\int_a^b f(x)g(x)\,dx\Big| \leq \Big(\int_a^b |f(x)|^2\,dx\Big)^{1/2}\Big(\int_a^b |g(x)|^2\,dx\Big)^{1/2}$$

$f,g \ = \int_a^b f(x)g(x)\,dx$ is an inner product on $C[a,b]$.

Theorem 4 *If* $\cdot,\cdot$ *is an inner product on* V*, then for* $x \in V$

$$\|x\| = \sqrt{x,x}$$

is a norm.

Positivity: $\|x\| = \sqrt{x, x} \geq 0$

Homogeneity: $\|ax\| = \sqrt{ax, ax} = \sqrt{a^2 x, x} = |a| \cdot \|x\|$

Triangle inequality:

$$\begin{aligned} \|x+y\|^2 &= x+y, x+y = x, x + 2x, y + y, y \\ &= \|x\|^2 + 2x, y + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \, \|y\| + \|y\|^2 \quad \text{[Cauchy-Schwartz]} \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

Therefore, $\|x+y\| \leq \|x\| + \|y\|$.

The triangle inequality for the usual Euclidean distance also follows this theorem.

1.5 Complex Case

Inner products and norms can be defined on vector spaces over , but definitions must be modified.

Definition 5 *A* hermitian *inner product,* $\cdot, \cdot : V \times V \to$, *satisfies*

1. $x, x \in$ *and* $x, x \geq 0$ *with "=" iff* $x = 0$.
2. $x, y = \overline{y, x}$
3. $ax + by, z = ax, z + by, z$ *and* $z, ax + by, z = \bar{a}z, x + \bar{b}z, y$

$\forall \, x, y, z \in V, \; a, b \in.$

A *norm* can be defined by $\|x\| = \sqrt{x, x}$ which satisfies $\|ax\| = |a| \cdot \|x\|$ where $|a| = \sqrt{a \cdot \bar{a}}$. Moreover the Cauchy-Schwartz inequality holds, $|x, y| \leq \|x\| \cdot \|y\|$, which implies the triangle inequality for the norm. Proofs are omitted.