1 Topology of Metric Spaces

1.1 Open Sets

Recall that a metric space is a set M with a distance function $d : M \times M \rightarrow$ satisfying

1. $d(x, y) \geq 0$ with "=" iff $x = y$ (*positivity*)
2. $d(x, y) = d(y, x)$ (*symmetry*)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*)

$\forall\, x, y, z \in M$.

1.1.1 Examples

1) ^{n} with norm $\|x\|_p = \left(\sum x_i^p\right)^{1/p}$ or $\|x\|_\infty = \max\{|x_i|\}$.

2) $C[a, b]$ with norm $\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$ or $\|f\|_\infty = \sup |f(x)|$

3) Let M be any set and for $x, y \in M$ define

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Then $d(x, y)$ clearly satisfies positivity and symmetry and it is not hard to show that it also satisfies the triangle inequality. This metric is called the "discrete metric" on M. The only point that is "close" to a given point $x \in M$ is x itself!

Definition 1 *The set* $B(x, r) = \{y \in M \mid d(x, y) < r\}$ *is called the* open ball *around* x *with* radius r.

Open balls play the same role as open intervals in . Many definitions and theorems for carry over to metric spaces by replacing the words 'open interval' with 'open ball.'

Definition 2 *A subset* $A \subset M$ *is* open *if* $\forall\, x \in A$, $\exists\, r > 0$ *such that* $B(x, r) \subset A$.

An arbitrary open set in a metric spaces does not usually have a structure that is simple to describe, like an open set in (the countable disjoint union of open intervals).

Definition 3 *A* metric subspace *of M is any subset M' with distance function of M restricted to M'.*

Theorem 1 *Let M' be a metric subspace of M. Then*

$$A' \text{ is open in } M' \iff \exists\, A \text{ open in } M \text{ such that } A' = A \cap M'$$

($\Rightarrow$) If A' is open in M', then $\forall\, x \in A'$, $\exists\, r > 0$ such that

$$B'(x,r) = \{y \in M' \mid d(x,y) < r\} \subset A'$$

Consider

$$B(x,r) = \{y \in M \mid d(x,y) < r\}$$

Then $B'(x,r) = B(x,r) \cap M'$. Let

$$A = \bigcup_{x \in A'} \left\{ B(x,r) \mid B'(x,r) \subset A' \right\}$$

Then A is open and

$$\begin{aligned} A \cap M' &= \bigcup_{x \in A'} \left\{ B(x,r) \cap M' \mid B'(x,r) \subset A' \right\} \\ &= \bigcup_{x \in A'} \left\{ B'(x,r) \mid B'(x,r) \subset A' \right\} = A' \end{aligned}$$

($\Leftarrow$) Let A be an open subset of M and let $x \in A' = A \cap M'$. By definition, $\exists\, B(x,r) \subset A$, so $B'(x,r) = B(x,r) \cap M' \subset A \cap M' = A'$. Therefore, A' is open.

Corollary 2 *Suppose M' is open in M. Then A' is open in M' $\iff$ A' is open in M.*

Theorem 3 *In any metric space:*

1. *The union of any number of open sets is open.*
2. *The intersection of a finite number of open sets is open.*

1. Let $x \in \bigcup U_\alpha$ where each U_α is open. Then $x \in U_\alpha$ for some α, so $\exists\, B(x,r) \subset U_\alpha$. Then $B(x,r) \subset \bigcup U_\alpha$, proving that the union is open.

2. Let $x \in U_1 \cap \ldots \cap U_n$ where each U_i is open. For each i there is a $B(x, r_i) \subset U_i$. Let $r = \min\{r_i\}$. Then $B(x,r) \subset B(x,r_1) \cap \cdots \cap B(x,r_n) \subset U_1 \cap \ldots \cap U_n$, which proves that the intersection is open.

A *topological space* is a set X with a collection of open subsets (including $\emptyset$ and X) that satisfying the two statements of the theorem. Many of our proofs for metric spaces would also apply more generally to topological spaces, but we do not need this level of generality.

1.2 Closed Sets

Let M be a metric space with distance function $d(x, y)$.

Definition 4 *1. We say that $x \in M$ is the* limit *of a sequence $\{x_n\} \subset M$, written $x = \lim x_n$, if*

$$\forall\, 1/m, \exists\, N \text{ such that } d(x_n, x) < 1/m, \forall\, n \geq N$$

2. We say that $x \in M$ is a limit point *of a set $A \subset M$ if $\exists\, \{x_n\} \subset A$ such $x = \lim x_n$.*

Note that an equivalent condition for x to be a limit point of A is

$$B(x, r) \cap A \neq \emptyset \quad \forall\, r > 0$$

Definition 5 *1. A set $A \subset M$ is* closed *if it contains all of its limit points.*

2. The closure *of a set $A \subset M$, denoted by $\overline{A}$, is the union of A and all of its limit points.*

Theorem 4 *$A \subset M$ is closed $\iff$ A^c is open.*

($\Rightarrow$) Let $y \in A^c$. Then y is not a limit point of A (closed). So $\exists\, B(y, r) \subset A^c$ (otherwise every open ball around y intersects A which implies y is a limit point of A). Therefore, A^c is open.

($\Leftarrow$) Let x be a limit point of A. If $x \in A^c$ then $\exists\, B(x, r) \subset A^c$ (open). Contradiction! Therefore, $x \in A$.

Theorem 5 *In any metric space:*

1. *The union of a finite number of closed sets is closed.*
2. *The intersection of any number of closed sets is closed.*

These statements follow from Theorems **??**, **??**, and the following identities:

$$(U_1 \cup \ldots \cup U_n)^c = U_1^c \cap \ldots \cap U_n^c \text{ and } (\bigcap U_\alpha)^c = \bigcup U_\alpha^c$$

1.3 Completeness

Let M be a metric space with distance function $d(x, y)$.

Definition 6 *A sequence* $\{x_n\} \subset M$ *is a* Cauchy *sequence if* $\forall\, 1/m$, $\exists\, N$ *such that* $d(x_j, x_k) < 1/m$, $\forall\, j, k \geq N$.

Theorem 6 *If* $\{x_n\}$ *converges, then* $\{x_n\}$ *is Cauchy.*

Let $x = \lim x_n$. Given $1/m$, $\exists\, N$ such that $d(x_n, x) < 1/(2m)$, $\forall\, n \geq N$. Therefore, by the triangle inequality,

$$d(x_j, x_k) \leq d(x_j, x) + d(x, x_k) < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m} \quad \forall\, j, k \geq N$$

Example: The converse is *not* true for a general metric space. Consider as a metric subspace of . The sequence

$$x_n = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^n \frac{1}{2n+1} \in$$

converges to $\pi/4 \in$, so the sequence is Cauchy but does not converge to a point in .

Definition 7 *A metric space is* complete *if every Cauchy sequence converges.*

Example: ^{n} with Euclidean distance is a complete metric space. If a sequence $x_j = (x_{j1}, \ldots, x_{jn}) \in^n$ is Cauchy, then each coordinate x_{jk} (for fixed k) defines a Cauchy sequence in , and so converges, $x_{jk} \to a_k$. Thus $x_j \to a = (a_1, \ldots, a_n) \in^n$.

Theorem 7 $C[a, b]$ *with the norm* $\|f\|_\infty = \sup\{|f(x)| \mid x \in [a, b]\}$ *is a complete metric space.*

Let $\{f_n\} \subset C[a, b]$ be a Cauchy sequence. Given $1/m$, $\exists\, N$ such that $\|f_j - f_k\|_\infty < 1/m$, $\forall\, j, k \geq N$. This implies that

$$|f_j(x) - f_k(x)| < 1/m \quad \forall\, x \in [a, b], \forall\, j, k \geq N$$

Thus, $\{f_n(x)\}$ is a Cauchy sequence in and so converges, say $f_n(x) \to f(x)$. Letting $k \to \infty$ in the previous inequality, we get

$$|f_j(x) - f(x)| \leq 1/m \quad \forall\, x \in [a, b], \forall\, j \geq N$$

Therefore f_n converges to f uniformly, and by Theorem **??**, $f \in C[a, b]$. Hence, $C[a, b]$ is complete.

Remarks:

1) As the proof demonstrates, convergence in the sup-norm, $\|f_n - f\|_\infty \to 0$, is the same as uniform convergence of functions, $f_n \to f$.

2) $C[a, b]$ is *not* complete with respect to other norms. The example of section 2.1 gives a sequence of continuous functions on $[0, 2]$ that converges pointwise to a discontinuous function on $[0, 2]$. These functions form a Cauchy sequence with respect to the norm $\|f\|_1 = \int_0^2 |f(x)| dx$ but do not converge to a function in $C[a, b]$.

1.4 Compactness

Let M be a metric space with distance function $d(x, y)$. Our previous notion of compactness carries over directly to metric spaces.

Definition 8 *A set $A \subset M$ is* compact *if every open cover of A has a finite subcover.*

If M is a normed vector space then a natural definition for a set A to be *bounded* is that there exists a constant K such that

$$\|x\| \leq K \quad \forall x \in A$$

This concept can be extended to general metric spaces by rewriting the boundedness condition in terms of the metric: If $\|x\| \leq K$, $\forall x \in A$, then $d(x, y) = \|x - y\| \leq \|x\| + \|y\| \leq 2K$, $\forall x, y \in A$. Conversely, if $d(x, y) \leq K$, $\forall x, y \in A$, then fix $z \in A$ and let $L = \|z\|$. Then $\|x\| =\leq \|x - z\| + \|z\| \leq K + L$, $\forall x \in A$.

Definition 9 *The* diameter *of a set A is*

$$d(A) = \sup_{x,y \in A} d(x, y)$$

The set A is bounded *if $d(A) < \infty$.*

If $A \subset M$ is compact, then A is bounded.

The open balls $B(a, 1)$, for $a \in A$, cover A, so there is a finite number, $a_1, \ldots, a_n \in A$, such that $A \subset B(a_1, 1) \cup \cdots \cup B(a_n, 1)$. Let $M = \max d(a_i, a_j)$, $i, j = 1, \ldots, n$. For any $x, y \in A$ there exist a_i and a_j such that $x \in B(a_i, 1)$ and $y \in B(a_j, 1)$. By the triangle inequality,

$$d(x, y) \leq d(x, a_i) + d(a_i, a_j) + d(a_j, y) \leq 1 + M + 1$$

which proves that $d(A) \leq M + 2$.

For the next proof it is convenient to have a notion of the distance of a point x to a set A:

$$d(x, A) = \inf_{y \in A} d(x, y)$$

Any compact metric space has a countable dense subset.

We may assume the metric space M has an infinite number of points (otherwise M itself is a countable dense subset).

Let z_1 be any point in M and define a sequence $\{z_n\}$ inductively as follows: Let R_n be the maximum possible distance in M from the set $\{z_1, \ldots, z_n\}$.

$$R_n = \sup_{x \in M} d(x, \{z_1, \ldots, z_n\})$$

Note that $R_n < \infty$ by Proposition **??**. Then choose z_{n+1} to be any point in M satisfying

$$d(z_{n+1}, \{z_1, \ldots, z_n\}) \geq R_n/2$$

As more and more points are added, the maximum possible distance from those already chosen must decrease. In fact, it is clear that $d(x, \{z_1, \ldots, z_n\}) \geq d(x, \{z_1, \ldots, z_{n+1}\})$ so the distances are monotonic decreasing, $R_n \geq R_{n+1}$, and bounded below by 0. Let

$$R = \lim_{n \to \infty} R_n \geq 0$$

Note that

$$\begin{aligned} & d(z_{n+1}, \{z_1, \ldots, z_n\}) \geq R_n/2 \geq R/2 \quad \forall\, n \\ \Rightarrow \quad & d(z_{n+1}, z_j) \geq R/2 \quad \forall\, j < n+1 \\ \Rightarrow \quad & d(z_k, z_j) \geq R/2 \quad \forall\, j \neq k \end{aligned}$$

(Since $d(z_k, z_j) = d(z_j, z_k)$ we do not need to assume $j < k$.)

The compactness of M implies that $R = 0$. To see why this is true, suppose $R > 0$. Then the open balls $B(x, R/4)$, $x \in M$, cover M, so a finite number of them cover M. But then at least two of the infinite sequence, say z_j and z_k, must lie in the same open ball of radius $R/4$ and hence $d(z_j, z_k) < R/2$, contradicting the previous inequality. Therefore, $R = 0$.

Finally, we show that $\{z_n\}$ is dense in M by showing that if $x \in M$ is an arbitrary point, there is a subsequence of $\{z_n\}$ that converges to x. Since $\lim R_n = 0$, given any $m \in$, there is some $R_n < 1/m$. By the definition of R_n, the distance of *any* point of M to $\{z_1, \ldots, z_n\}$ is $\leq R_n$. In particular, $d(x, \{z_1, \ldots, z_n\}) < 1/m$ so $\exists\, z_{k(m)}$ (for some $1 \leq k(m) \leq n$) such that $d(x, z_{k(m)}) < 1/m$. The subsequence $\{z_{k(m)}\}$ obviously converges to x: Given any $1/m$, $d(x, x_{k(j)}) < 1/j \leq 1/m$, $\forall\, j \geq m$.

The proof of the following theorem closely parallels the proof given for compact sets of real numbers, Theorem **??**.

Theorem 8 *A set $A \subset M$ is compact if and only if every sequence $\{x_n\} \subset A$ has a limit point in A, i.e., has a subsequence that converges to a point in A.*

($\Rightarrow$) By contradiction. Assume $\exists\, \{x_n\} \subset A$ that has no limit points in A. Let C_k be the closure of the set $\{x_n \mid n \geq k\}$. Then the sets $B_k = M \setminus C_k$ are open and cover A, $A = \bigcup B_k$: If $x \in A$, then x cannot be a limit point of $\{x_n\}$ by assumption. Likewise, $x \neq x_n$, except for possibly a finite number of n, otherwise x would be a limit point of $\{x_n\}$. Thus, there is some N such that $x \neq x_n$, $\forall\, n \geq N$. Therefore $x \notin C_N$, and so $x \in B_N$.

Since A is compact, there is a finite subcover, say $A \subset B_{k_1} \cup \ldots \cup B_{k_t}$. The sets B_k are nested, so $B_{k_1} \cup \ldots \cup B_{k_t} = B_k$ where $k = \max\{k_1, \ldots, k_t\}$. But this means that $x_k, x_{k+1}, \ldots$ are not in A, a contradiction!

($\Leftarrow$) Let $= \{U_\alpha\}$ be an open cover of A. We must show A is contained in a finite union of U_α, $A \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_t}$.

Let $\{z_n\}$ be a countable dense set of points in M, see Proposition **??**. We can then find a countable subcover of that still covers A as follows. Consider the countable number of open sets $B(z_n, 1/m)$, $n, m \in$. For each $B(z_n, 1/m)$ choose *one* $U_\beta \in$ that contains $B(z_n, 1/m)$, if any. Let $' = \{U_\beta\}$ be the resulting countable subcollection. To see that $'$ still covers A, note that $x \in A \Rightarrow x \in U_\alpha$ for some α. Since U_α is open, it contains an open ball around x, say $B(x, r) \subset U_\alpha$. Let $1/m < r/2$. Since $\{z_n\}$ is dense in M, $\exists\, z_n$ such that $d(z_n, x) < 1/m$. But then $x \in B(z_n, 1/m) \subset B(x, r)$, so $\exists\, U_\beta \in'$ such that $x \in U_\beta$ and hence $'$ covers A.

We now have a countable subcover, say, $U_1, U_2, \ldots$. If we take n large enough, then $U_1, U_2, \ldots, U_n$ must already cover A. Suppose not. Then for each n, $\exists\, x_n \in A$ that is not contained in $U_1, \ldots, U_n$. By assumption, the sequence $\{x_n\}$ has a limit point $x \in A$. Thus $x \in U_k$ for some k. But by construction, U_k does not contain $x_k, x_{k+1}, \ldots$ contradicting the fact that any neighborhood of a limit point must contain an infinite number of points in the sequence.

Remarks:

1) The previous theorem shows that a compact metric space is complete: a Cauchy sequence can have at most one limit point and the theorem guarantees at least one limit point, so the sequence converges. Along with Theorem **??**, we have

$$\textit{Compact} \Rightarrow \textit{Complete and Bounded}$$

The converse of this statement is *not* true (see the example below).

2) "Completeness" in some sense plays the role of being "closed" in the abstract setting where being closed may have little meaning (e.g., example, any metric space is itself closed). The connection between completeness and being closed is best described in the following statement whose proof is left as an exercise for the reader:

> *Let M be a complete metric space. A subspace $A \subset M$ is complete if and only if it is closed.*

3) We know that ^{n} is complete, so closed subsets are the same as complete subsets. We also know that a subset of is compact if and only if it is closed and bounded. It is not hard to derive from this that a subset of ^{n} is compact if and only if it is closed (complete) and bounded.

Example: The complete metric space C[a,b] with the sup-norm distance function

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

contains subspaces that are closed (complete) and bounded, yet are not compact. Let $A = \{f_n(x)\} \subset C[0, 2]$ where

$$f_n(x) = \begin{cases} 0 & 0 \leq x < 1 - \frac{1}{n} \\ \frac{n}{2}(x-1) + \frac{1}{2} & 1 - \frac{1}{n} \leq x < 1 + \frac{1}{n} \\ 1 & 1 + \frac{1}{n} \leq x \leq 2 \end{cases}$$

(see the example and illustration in section 2.1). The set A is closed because it has no limit points in $C[0,2]$: The sequence f_n converges pointwise to a discontinuous function, so the convergence is not uniform, and therefore does not converge with respect to the sup-norm. The set A is also bounded because it has finite diameter

$$d(A) = \sup d(f_n, f_m) = 1/2$$

But A is not compact for the same reason it is "closed." The sequence $\{f_n\}$ has no convergent subsequence in A (or in $C[0,2]$).

We now give an interpretation of the Arzela-Ascoli Theorem **??** in terms of compactness in $C[a,b]$.

Theorem 9 *Let A be a subspace of the complete metric space $C[a,b]$ with sup-norm distance function. Then A is compact if and only if A is closed, bounded, and uniformly equicontinuous.*

($\Rightarrow$) Since $C[a,b]$ is complete, we know A is closed by Remark 2 above. It is bounded by Theorem **??**. To show that A is uniformly equicontinuous we need to show that given any $1/m$, $\exists\, 1/n$ such that $|x-y| < 1/n \Rightarrow |f(x) - f(y)| < 1/m$, $\forall\, f \in A$. Since A is covered by the open balls $B(f, 1/(3m))$ for $f \in A$, a finite number cover A, say $A \subset B(f_1, 1/(3m)) \cup \ldots B(f_t, 1/(3m))$. For each $i = 1, \ldots, t$, $\exists\, 1/n_i$ such that $|f_i(x) - f_i(y)| < 1/(3m)$ whenever $|x-y| < 1/n_i$ (see Theorem **??**). Let $n = \max\{n_1, \ldots, n_t\}$. Assume $|x-y| < 1/n$ and let $f \in A$ be arbitrary. Then $f \in B(f_i, 1/(3m))$ for some i, so $|f(x_i) - f(x)| < 1/(3m)$, $\forall\, x \in [a,b]$. Therefore,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &< \frac{1}{3m} + \frac{1}{3m} + \frac{1}{3m} = \frac{1}{m} \end{aligned}$$

($\Leftarrow$) By Theorem **??**, we must show that any sequence of functions $\{f_n\} \subset A$ has a convergent subsequence. Since A is bounded, $\exists\, K$ such that $\|f_n\| = \sup_{x\in[a,b]} |f_n(x)| \leq K$, $\forall n$, so $\{f_n\}$ is uniformly bounded. The sequence is uniformly equicontinuous by assumption, so by Theorem **??**, there is a subsequence, $\{f_{n'}\}$ that converges uniformly to $f \in C[a,b]$. In particular, $\|f_n - f\|_\infty \to 0$. Because A is closed, $f \in A$.