1 Existence and Uniqueness of Solutions

1.1 Systems of Ordinary Differential Equations

Let $x : [a.b] \to^n$ be a vector function defined on an interval of real numbers. Denote by $x^{(m)}(t)$ the m-th derivative of $x(t)$ with respect to t. Suppose $G : [a, b] \times D \to^n$ is a continuous function for some open subset $D \subset^{nm}$. An equation of the form

$$x^{(m)}(t) = G(t, x(t), x'(t), \ldots, x^{(m-1)}(t))$$

is called a *system of ordinary differential equations of order m.*

By increasing the number of unknowns and equations, any m-th order system can be written as a first order system, as follows. Define

$$v_1(t) = x(t),\ v_2(t) = x'(t), \ldots,\ v_m(t) = x^{(m-1)}(t)$$

Then the equivalent first order system is given by the equations

$$v_1'(t) = v_2(t),\ v_2'(t) = v_3(t), \ldots,\ v_{m-1}'(t) = v_m(t),$$
$$v_m'(t) = G(t, v_1(t), \ldots, v_m(t))$$

or more compactly as

$$v'(t) = H(t, v(t))$$

where $v : [a, b] \to^{nm}$ is $v(t) = (v_1(t), \ldots, v_m(t))$ and $H : [a, b] \times D \to^{nm}$ is $H(t, v(t)) = (v_2(t), \ldots, v_m(t), G(t, v_1(t), . .$

Example: Consider the spring-mass system shown below. The two masses move on a frictionless surface under the influence of external forces $F_1(t)$ and $F_2(t)$ and they are also constrained by the three springs whose constants are k_1, k_2, and k_3.

From Newton's law of motion and Hooke's law for springs, we obtain the following equations for the coordinates x_1 and x_2 of the two masses

$$\begin{aligned} m_1 x_1''(t) &= -k_1 x_1(t) + k_2(x_2(t) - x_1(t)) + F_1(t) \\ m_2 x_2''(t) &= -k_3 x_2(t) - k_2(x_2(t) - x_1(t)) + F_2(t) \end{aligned}$$

To put this in vector form, $x''(t) = G(t, x(t), x'(t))$, we let

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

and

$$G(t, x(t), x'(t)) = \begin{bmatrix} (k_2 x_2(t) - (k_1 + k_2) x_1(t) + F_1(t))/m_1 \\ (k_2 x_1(t) - (k_2 + k_3) x_2(t) + F_2(t))/m_2 \end{bmatrix}$$

The equivalent first order system is $v'(t) = H(t, v(t))$, where

$$v(t) = \begin{bmatrix} v_{11}(t) \\ v_{12}(t) \\ v_{21}(t) \\ v_{22}(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_1'(t) \\ x_2'(t) \end{bmatrix}$$

and

$$H(t, v(t)) = \begin{bmatrix} v_{21} \\ v_{22} \\ (k_2 v_{12}(t) - (k_1 + k_2) v_{11}(t) + F_1(t))/m_1 \\ (k_2 v_{11}(t) - (k_2 + k_3) v_{12}(t) + F_2(t))/m_2 \end{bmatrix}$$

The process of solving a differential equation is essentially a form of integration and always involves undetermined constants as part of the answer. In order to get a unique solution we must impose extra conditions that determine the value of these constants. Typically *initial conditions* are specified, that is, a solution to $x'(t) = G(t, x(t))$ must satisfy $x(t_0) = x_0$ for some given $t_0 \in [a, b]$ and $x_0 \in D$. A system of differential equations with an initial condition is called an *initial value problem.*

We shall prove, under mild assumptions about the function G, that the solution to an initial value problem always exists and is unique. Our proof is based on the Contractive Mapping Principle which requires converting the initial value problem into an equivalent integral equation, then using the integral equation to define a contractive mapping of a certain complete metric space into itself. The solution will be the fixed point of the contractive mapping.

1.2 Integral Equations

We say that a function f is C^k if f has continuous derivatives up to order k. We write $C^k(D)$ to denote the vector space of C^k functions with common domain D. We sometimes need to specify a common range as well, so we let $C^k(D, E)$ denote the set of C^k functions $f : D \to E$.

For example, $C^k([a, b],^n)$ denotes the vector space of continuous vector functions $f : [a, b] \to^n$ that have k continuous derivatives. Since derivatives of vector functions are taken component-wise, it follows that

$$C^k([a, b], R^n) \cong C^k[a, b] \times \cdots \times C^k[a, b] \quad (n \text{ times})$$

The translation of an initial value problem to an integral equation follows immediately from the Fundamental Theorem of Calculus for vector functions.

Let $G : [a, b] \times D \to^n$ be a continuous function defined on some neighborhood of $(t_0, x_0) \in \times^n$. If $x(t) : [a, b] \to^n$ is C^1 and $x(t)$ satisfies the initial value problem

$$x'(t) = G(t, x(t)), \quad x(t_0) = x_0 \tag{1}$$

then $x(t)$ satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} G(s, x(s))\, ds \tag{2}$$

Conversely, if $x(t) : [a, b] \to^n$ is continuous and $x(t)$ satisfies (**??**), then $x(t)$ is C^1 and $x(t)$ satisfies (**??**).

The first statement follows by integrating the differential equation to get

$$x(t) - x(t_0) = \int_{t_0}^{t} x'(s)\, ds = \int_{t_0}^{t} G(s, x(s))\, ds$$

The second statement follows by differentiating the integral equation to get

$$\frac{d}{dt} x(t) = \frac{d}{dt}\Big(x_0 + \int_{t_0}^{t} G(s, x(s))\, ds\Big) = G(t, x(t))$$

Notice that the integral equation has the initial condition "built-in." Also, we need only assume $x(t)$ is continuous for the integral to exist, yet any solution of the integral equation gets "promoted" to having a continuous derivative.

Now we describe the type of complete metric space on which we shall construct a contractive mapping. The sup-norm on $C([a, b],^n)$ is

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$$

where $|f(x)|$ is the Euclidean norm of the vector $f(x)$. Convergence with respect to this norm is equivalent to component-wise convergence with respect to the sup-norm on $C[a, b]$.

Let B be a closed subset of ^{n}. Then the metric space $C([a, b], B)$ is complete with respect to the sup-norm.

Let $\{f_j\} \subset C([a, b], B)$ be a Cauchy sequence. Then $\|f_j - f_k\|_\infty \to 0$ implies $\|f_{ji} - f_{ki}\|_\infty \to 0$ where f_{ji} are the components of $f_j = (f_{j1}, \ldots, f_{jn})$, and hence $\{f_{ji}\} \subset C[a, b]$ is a Cauchy sequence for each $i = 1, \ldots, n$. Since $C[a, b]$ is complete by Theorem **??**, $f_{ji} \to f_i \in C[a, b]$, so $f_j \to f = (f_1, \ldots, f_n) \in C([a, b],^n)$. Finally, Since B is closed, $f(x) = \lim f_j(x) \in B$, so $f \in C([a, b], B)$.

The next two lemmas take care of some technical details. The first is an extension of a standard inequality for continuous functions, to continuous vector functions.

[Minkowski's Inequality] If $v : [a, b] \to^n$ is a continuous function, then

$$\Big|\int_a^b v(s)\, ds\Big| \leq \int_a^b |v(s)|\, ds$$

The statement follows from the triangle inequality for the Euclidean metric. Integration of a vector function is done component-wise,

$$\int_a^b v(s)\,ds = \Big(\int_a^b v_1(s)\,ds, \ldots, \int_a^b v_n(s)\,ds\Big)$$

and for each component

$$\int_a^b v_i(s)\,ds = \sup\Big\{\int_a^b t_i(s)\,ds \;\Big|\; t_i \text{ (step fn)} \leq v_i\Big\} = \lim_{j\to\infty}\int_a^b t_{ij}(s)\,ds$$

for some sequence $\{t_{ij}\}$ of step functions $\leq v_i$. Recall that the integral of a step function is a finite sum,

$$\int_a^b t_{ij}(s)\,ds = \sum_k t_{ij}(s_k)\Delta s_k$$

Therefore, by using a common refinement for the partitions of the step functions $t_{1j}, \ldots, t_{nj}$, for each j, we get

$$\begin{aligned}
\Big|\int_a^b v(s)\,ds\Big| &= \lim_{j\to\infty}\Big|\sum_k (t_{1j}(s_k), \ldots, t_{nj}(s_k))\Delta s_k\Big| \\
&\leq \lim_{j\to\infty}\sum_k |(t_{1j}(s_k), \ldots, t_{nj}(s_k))|\Delta s_k \quad \text{[T.I.]} \\
&\leq \sup\Big\{\int_a^b u(s)\,ds \;\Big|\; u \text{ (step fn)} \leq |v|\Big\} \\
&= \int_a^b |v(s)|\,ds
\end{aligned}$$

Finally, we establish a crucial regularity property of G that follows from the continuity of the derivatives of G.

[Lipschitz Condition] Let D be open an open subset of ^{n} and let $G : [a, b] \times D \to^n$ be a C^1 function. Then for any $x_0 \in D$ and any open ball $B = B(x_0, \epsilon)$ such that $\overline{B} \subset D$, there exists an $M > 0$ such that

$$|G(t, x) - G(t, y)| \leq M|x - y|, \quad \forall\, (t, x), (t, y) \in [a, b] \times \overline{B}$$

Let $(t, x), (t, y) \in [a, b] \times \overline{B}$ be arbitrary. Since the ball $\overline{B}$ is convex, the line segment from x to y is contained in $\overline{B}$,

$$x + s(x - y) \in \overline{B}, \quad \forall\, s \in [0, 1]$$

Consider the continuous function $g : [0, 1] \to^n$ defined by

$$g(s) = G(t, x + s(y - x))$$

Then

$$G(t, y) - G(t, x) = g(1) - g(0) = \int_0^1 g'(s)\,ds$$

By the Chain Rule,

$$g'(s) = \sum_{k=1}^{n} (y_k - x_k) \frac{\partial G}{\partial x_k}(t, x + s(y - x))$$

Let m be the maximum of all the continuous functions $|\partial G/\partial x_k|$, $k = 1, \ldots, n$, on the compact set $[a, b] \times \overline{B}$ and let $M = m \cdot n$. Then

$$\begin{aligned}
|G(t, y) - G(t, x)| &= \Big| \int_0^1 g'(s)\, ds \Big| \\
&\leq \int_0^1 |g'(s)|\, ds \quad \text{[Lemma ??]} \\
&\leq \sum_{k=1}^{n} |y_k - x_k| \int_0^1 \Big| \frac{\partial G}{\partial x_k}(t, x + s(y - x)) \Big|\, ds \\
&\leq m \sum_{k=1}^{n} |y_k - x_k| \\
&\leq m \cdot n \cdot \|y - x\|_\infty \\
&\leq M \cdot |x - y|
\end{aligned}$$

1.3 Existence and Uniqueness of Solutions

Now we come to the proof of the existence and uniqueness of a solution to an initial value problem. The basic idea is to show that the mapping of continuous vector functions defined by $Tx(t) = x_0 + \int_{t_0}^t G(s, x(s))\, ds$ is, under certain restrictions, a *contractive mapping*. The Contractive Mapping Principle implies that T has a unique fixed point, say $T\phi(t) = \phi(t)$. The function $\phi(t)$ then satisfies the integral equation $\phi(t) = T\phi(t) = x_0 + \int_{t_0}^t G(s, \phi(s))\, ds$, and therefore the initial value problem $\phi'(t) = G(t, \phi(t))$, $\phi(t_0) = x_0$.

Theorem 1 (Existence and Uniqueness) *Let $G : [a, b] \times D \to^n$ be a C^1 function defined on some neighborhood of $(t_0, x_0) \in \times^n$. Then there is a subinterval $[c, d] \subset [a, b]$ containing t_0 such that the initial value problem*

$$x'(t) = G(t, x(t)), \quad x(t_0) = x_0$$

has a unique solution defined on the interval $[c, d]$.

Let $B = B(x_0, \epsilon)$ be such that $\overline{B} \subset D$. By Lemma **??**, $\exists\, M > 0$ such that

$$|G(t, x) - G(t, y)| \leq M|x - y|, \quad \forall\, (t, x), (t, y) \in [a, b] \times \overline{B}$$

Let $\|G\|_\infty$ be the maximum of $|G(t, x)|$ on $[a, b] \times \overline{B}$. Fix $0 < r < 1$ and let $[c, d]$ be the interval $[t_0 - \epsilon_0, t_0 + \epsilon_0]$ where

$$\epsilon_0 = \min\{\epsilon/\|G\|_\infty, r/M\}$$

For a continuous function $x : [c, d] \to \overline{B}$ define

$$Tx(t) = x_0 + \int_{t_0}^t G(s, x(s))\, ds$$

Then

$$|Tx(t) - x_0| = \Big| \int_{t_0}^{t} G(s, x(s))\, ds \Big| \le \|G\|_\infty |t - t_0| \le \epsilon$$

since $|t - t_0| \le \epsilon_0 \le \epsilon/\|G\|_\infty$. Thus, $Tx : [c, d] \to \overline{B}$ showing that T is a mapping of the metric space $C([c, d], \overline{B})$ into itself. Moreover, by Lemma **??**, $C([c, d], \overline{B})$ with the sup-norm is a complete metric space.

We now show that T is a contractive mapping. If $x, y \in C([c, d], \overline{B})$ then

$$\begin{aligned}
\|Tx - Ty\|_\infty &= \Big\| x_0 + \int_{t_0}^{t} G(s, x(s))\, ds - x_0 - \int_{t_0}^{t} G(s, y(s))\, ds \Big\|_\infty \\
&= \sup_{t \in [c,d]} \Big| \int_{t_0}^{t} G(s, x(s)) - G(s, y(s))\, ds \Big| \\
&\le \sup_{t \in [c,d]} \Big| \int_{t_0}^{t} |G(s, x(s)) - G(s, y(s))|\, ds \Big| \quad \text{[Lemma ??]} \\
&\le \sup_{t \in [c,d]} \Big| \int_{t_0}^{t} M|x(s) - y(s)|\, ds \Big| \quad \text{[Lemma ??]} \\
&\le M\|x - y\|_\infty \sup_{t \in [c,d]} |t - t_0| = M\|x - y\|_\infty \epsilon_0 \\
&\le r\|x - y\|_\infty
\end{aligned}$$

since $\epsilon_0 \le r/M$.

By the Contractive Mapping Principle, Theorem **??**, T has a unique fixed point $\phi \in C([c, d], \overline{B})$. Therefore, $\phi(t)$ is the unique solution of the integral equation

$$\phi(t) = T\phi(t) = x_0 + \int_{t_0}^{t} G(s, \phi(s))\, ds$$

and hence, by Lemma **??**, $\phi(t)$ is the unique solution of the initial value problem

$$\phi'(t) = G(t, \phi(t)), \quad \phi(t_0) = x_0$$

1.4 Picard Iteration

Let us apply the method of the previous proof to find a solution to the initial value problem

$$x'(t) = 2t(1 + x(t)), \quad x(0) = 0$$

Here $G(t, x) = 2t(1 + x)$ has continuous first derivatives and the Lipschitz Condition is easy to determine. If $|t| \le R$, then

$$|G(t, x) - G(t, y)| = |2t + 2tx - 2t - 2ty| = 2|t| \cdot |x - y| \le 2R|x - y|$$

So $M = 2R$ independent of x and y. The maximum of G on $[-R, R] \times [-\epsilon, \epsilon]$ is

$$\|G\|_\infty = \sup |2t(1 + x)| = 2R(1 + \epsilon)$$

The proof of the theorem says that for some fixed $0 < r < 1$ the mapping

$$Tx(t) = 0 + \int_0^t 2s(1 + x(s))\, ds$$

is contractive if we restrict t to the interval $[-\epsilon_0, \epsilon_0]$ where

$$\epsilon_0 = \min\left\{\frac{\epsilon}{\|G\|_\infty}, \frac{r}{M}\right\} = \min\left\{\frac{\epsilon}{2R(1+\epsilon)}, \frac{r}{2R}\right\} < \frac{1}{2R}$$

For the largest interval of t we should choose $R = 1/\sqrt{2}$.

The solution $\phi(t)$ of the initial value problem is the fixed point of T which can be found, according to the Contractive Mapping Principle, by iterating the mapping T on any starting function $\phi_0(t)$. Let $\phi_0(t) = 0$ and for $n \geq 1$ define recursively

$$\phi_n(t) = T\phi_{n-1}(t) = \int_0^t 2s(1 + \phi_{n-1}(s))\, ds$$

Then $\phi_n(t) = T^n\phi_0(t)$ and

$$\phi(t) = \lim_{n\to\infty} T^n\phi_0(t) = \lim_{n\to\infty} \phi_n(t)$$

We obtain

$$\begin{aligned}
\phi_1(t) &= T\phi_0(t) = \int_0^t 2s(1+0)\, ds = t^2 \\
\phi_2(t) &= T\phi_1(t) = \int_0^t 2s(1+s^2)\, ds = t^2 + \frac{t^4}{2} \\
\phi_3(t) &= T\phi_2(t) = \int_0^t 2s(1+s^2+\frac{s^4}{2})\, ds = t^2 + \frac{t^4}{2} + \frac{t^6}{3\cdot 2}
\end{aligned}$$

These equations suggest that

$$\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!}$$

which can be proved by induction:

$$\begin{aligned}
\phi_n(t) &= \int_0^t 2s(1 + \phi_{n-1}(s))\, ds \\
&= \int_0^t 2s\left(1 + s^2 + \cdots + \frac{s^{2(n-1)}}{(n-1)!}\right) ds \\
&= t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!}
\end{aligned}$$

The solution to the initial value problem is thus

$$\phi(t) = \lim_{n\to\infty} \phi_n(t) = \sum_{n=1}^{\infty} \frac{t^{2n}}{n!} = e^{t^2} - 1$$

After finding the limit we see that it converges for all t, and hence the solution is also valid for all t, although the *a priori* restriction $|t| \leq 1/\sqrt{2}$ is needed to invoke the Contractive Mapping Principle.

1.5 Exercises

1. Consider the initial value problem

$$x'(t) + x(t)^2 = 0, \quad x(0) = x_0 > 0$$

   a) Show that $x(t) = x_0/(1 + x_0 t)$ is a solution for $t > -1/x_0$.

   b) The proof of Theorem **??** asserts that a solution exists on an interval $[x_0 - \epsilon_0, x_0 + \epsilon_0]$ where

$$\epsilon_0 = \min\left\{\frac{\epsilon}{\|G\|_\infty}, \frac{r}{M}\right\}$$

   Calculate ϵ_0 for this equation and verify that $x_0 - \epsilon_0 > -1/x_0$.

2. Find two distinct solutions of the initial value problem

$$x'(t) = x(t)^{1/3}, \quad x(0) = 0$$

   Explain how this is compatible with the uniqueness of solutions asserted by Theorem **??**.

3. Use Picard Iteration (the Contractive Mapping Principle) to compute the solution of the initial value problem

$$x'(t) = tx(t) + 1, \quad x(0) = 0$$

   a) Let $\phi_0(t) = 0$ and for $n \geq 1$ compute recursively

$$\phi_n(t) = T\phi_{n-1}(t) = 0 + \int_0^t (s\phi_{n-1}(t) + 1)\, ds$$

   b) Find $\phi(t) = \lim_{n\to\infty} \phi_n(t)$.