1 The Concept of Measure

The purpose of the Lebesgue integral is to enlarge the class of functions for which $\int_a^b f(x)dx$ has a well-defined value. The Riemann integral works well for continuous functions and uniform limits: if $f_n \in C[a,b]$ and $f_n \to f$ uniformly then $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$. It also works for bounded functions with a finite number of discontinuities but has problems with functions like Dirichlet's function,

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

Let $\{r_1, r_2, \ldots\} = \cap[a,b]$ be an enumeration of the rational numbers in $[a,b]$. Define

$$f_n(x) = \begin{cases} 1 & x = r_1, r_2, \ldots, r_n \\ 0 & \text{otherwise} \end{cases}$$

Then f_n is integrable ($\int_a^b f_n(x)\,dx = 0$) and for each $x \in [a,b]$, $f_n(x) \to f(x)$, and but f is not Riemann-integrable. The Lebesgue integral can handle functions and limits like this.

Lebesgue's own description of his method was in terms of a parable. A merchant could add up the day's receipts by

1. adding them in the order received, say,

$$5 + 10 + 1 + 1 + 25 + 5 + 10 + 50 + 25 + 10 = 142$$

   or

2. sorting the values first and adding them by type,

$$2 \times 1 + 2 \times 5 + 3 \times 10 + 2 \times 25 + 1 \times 50 = 142$$

The Riemann integral is like (1); it adds values of $f(x)$ in the order they occur by partitioning the *domain*, $\{x_0, x_1, \ldots, x_n\}$, and calculating the integral through approximations of the form

$$\sum_{j=1}^{n} f(x_j)(x_j - x_{j-1})$$

The Lebesgue integral is like (2); it sorts the values of $f(x)$ by partitioning the *range*, $\{y_0, y_1, \ldots, y_n\}$, and calculating the integral through approximations of the form

$$\sum_{j=1}^{n} y_j |A_j|$$

where

$$A_j = \{x \in [a,b] \mid y_{j-1} < f(x) \le y_j\}$$

and $|A_j|$ is the *size* of A_j. The hard part of the theory is arriving at a precise notion of $|A_j|$, and this leads to the concept of *Lebesgue measure*.

1.1 Properties of Length

Any method of measuring the size of sets of real numbers should be an extension of the concept of the length of an interval.

Definition 1 *If* $I = (a,b)$, $[a,b]$, $(a,b]$, *or* $[a,b)$, *then the* length *of* I *is* $|I| = b - a$. *If* $a = -\infty$ *or* $b = \infty$, *then* $|I| = \infty$. *If* $a = b$ *or* $I = \emptyset$, *then* $|I| = 0$.

The following properties of length are easy to verify.

Additivity: If $I = I_1 \cup \ldots \cup I_n$ (disjoint intervals), then $|I| = \sum_{j=1}^{n} |I_j|$.

Subadditivity: If $I \subset I_1 \cup \ldots \cup I_n$ (not necessarily disjoint intervals), then $|I| \leq \sum_{j=1}^{n} |I_j|$.

It is tempting to extend these properties to arbitrary unions, but this cannot work: If $I = [0,1]$ and $I_c = [c,c]$, then

$$I = \bigcup_{c \in [0,1]} I_c$$

but $|I_c| = 0$ and $|I| = 1$. Countable unions, on the other hand, are more tractable. In fact, the following holds.

σ-Additivity: If $I = \bigcup_{j=1}^{\infty} I_j$ (disjoint intervals), then $|I| = \sum_{j=1}^{\infty} |I_j|$.

If I is any interval then

$$|I| = \inf \left\{ \sum_{j=1}^{\infty} |I_j| \;\middle|\; I \subset \bigcup_{j=1}^{\infty} I_j \text{ (intervals)} \right\}$$

If $I_1 = I$ and $I_j = \emptyset$ for $j > 1$, then $I = \bigcup_{j=1}^{\infty} I_j$ and $|I| = \sum_{j=1}^{\infty} |I_j|$, so the infimum is *at most* $|I|$. It remains to show

$$I \subset \bigcup_{j=1}^{\infty} I_j \Rightarrow |I| \leq \sum_{j=1}^{\infty} |I_j|$$

If the I_j are not disjoint, we can shrink them to I'_j so they are and still maintain the same union. For example, if $I_j = [a,b]$, $I_k = [c,d]$ with $a < c < b < d$, then $I'_j = [a,c]$ and $I'_k = [c,d]$. Therefore,

$$I \subset \bigcup_{j=1}^{\infty} I'_j = \bigcup_{j=1}^{\infty} I_j$$

and $|I'_j| \leq |I_j|$. Now let $I''_j = I \cap I'_j$. Then

$$I = \bigcup_{j=1}^{\infty} I''_j \subset \bigcup_{j=1}^{\infty} I_j$$

and $|I''_j| \leq |I'_j| \leq |I_j|$. By σ-additivity,

$$|I| = \sum_{j=1}^{\infty} |I''_j| \leq \sum_{j=1}^{\infty} |I_j|$$

1.2 Measurable Sets

It turns out to be impossible to extend the definition of length $|I|$ from intervals I to arbitrary subsets $A \subset$ and still retain σ-additivity. It is possible to do this, however, for a special, but still very large, class of sets called *Borel sets*.

Definition 2 *A collection of sets is called a* field of sets *if it contains the empty set and is closed under the set operations of union, intersection, and complement.*

Thus if A, B are sets in a field , then $A \cup B$, $A \cap B$, and $A^c = \{x \in X \mid x \notin A\}$ are in . Here X is some fixed universe that contains all the sets in (and must be in itself).

Example: Let $X = [a, b]$ and let be the collection of finite unions of intervals contained in X. Then is a field.

Definition 3 *A field of sets is a* σ-field *if countable unions of sets in are again in ,*

$$A_j \in \Rightarrow \bigcup_{j=1}^{\infty} A_j \in$$

The previous example is a field but clearly not a σ-field. Given any field , we can construct a σ-field containing by adding all possible countable unions, their complements, intersections, etc., to . The process is somewhat technical to carry out and we shall skip the details.

Definition 4 *If is a field of sets, then the* σ-field generated by *, denoted* ${}_\sigma$*, is the intersection of all* σ*-fields containing .*

It is straightforward to check that ${}_\sigma$ is indeed a σ-field. It is clearly the smallest σ-field containing .

Definition 5 *Let be the field of finite unions of intervals in . Then the sets in* ${}_\sigma$ *are called the* Borel sets.

There is no really satisfactory description of the Borel sets, but the class is large enough to include any set describable in conventional mathematical terms. Our goal is to extend the length function from intervals to Borel sets while preserving σ-additivity. The extended length function is called *Lebesgue measure.*

1.3 Basic Properties of Measure

We first introduce the general concept of a measure and derive some of its properties.

Definition 6 *A* measure *is a function,* $\mu :\to \cup\{\infty\}$*, defined on a* σ*-field of sets , called the* measurable sets*, that satisfies*

1. Non-negativity: *If* A *is measurable then* $0 \leq \mu(A) \leq \infty$*, and* $\mu(\emptyset) = 0$*.*
2. σ-Additivity: *If* A_j *is a sequence of disjoint measurable sets, then*

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j)$$

Theorem 1 *For any measure* μ *the following properties hold:*

1. Monotonicity: *If* A *and* B *are measurable sets and* $A \subset B$*, then* $\mu(A) \leq \mu(B)$*.*
2. Continuity from below: *If* $A_1 \subset A_2 \subset A_3 \subset \ldots$ *with* A_j *measurable, then*

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \lim_{j\to\infty} \mu(A_j)$$

3. Conditional continuity from above: *If* $B_1 \supset B_2 \supset B_3 \supset \ldots$ *with* B_j *measurable and* $\mu(B_j)$ *finite, then*

$$\mu\Big(\bigcap_{j=1}^{\infty} B_j\Big) = \lim_{j\to\infty} \mu(B_j)$$

4. σ-Subadditivity: *If* A_j *is a sequence of measurable sets, then*

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) \leq \sum_{j=1}^{\infty} \mu(A_j)$$

1. $B \setminus A$ is measurable and $B = A \cup (B \setminus A)$ is a disjoint union, so by σ-additivity, $\mu(B) = \mu(A) + \mu(B \setminus A)$. Non-negativity then implies $\mu(B) \geq \mu(A)$.

2. The sets $B_j = A_j \setminus A_{j-1}$ are measurable and

$$\bigcup_{j=1}^{\infty} A_j = A_1 \cup B_2 \cup B_3 \ldots$$

is a disjoint union, so by σ-additivity,

$$\mu(\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \mu(A_1) + \mu(B_2) + \mu(B_3) + \ldots$$

Also, for any n,

$$A_n = A_1 \cup B_2 \cup B_3 \ldots \cup B_n$$

so

$$\mu(A_n) = \mu(A_1) + \mu(B_2) + \mu(B_3) + \ldots + \mu(B_n)$$

Therefore

$$\begin{aligned}
\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) &= \mu(A_1) + \mu(B_2) + \mu(B_3) + \ldots \\
&= \lim_{n\to\infty} \mu(A_1) + \mu(B_2) + \mu(B_3) + \ldots + \mu(B_n) \\
&= \lim_{n\to\infty} \mu(A_n)
\end{aligned}$$

3. The sets $B = \bigcap_{j=1}^{\infty} B_j$ and $A_j = B_j \setminus B_{j+1}$ are measurable and

$$B_1 = B \cup A_1 \cup A_2 \cup \ldots$$

is a disjoint union. By σ-additivity,

$$\mu(B_1) = \mu(B) + \mu(A_1) + \mu(A_2) + \ldots$$

Since $\mu(B)$ and $\mu(B_1)$ are finite we may write

$$\mu(B_1) - \mu(B) = \mu(A_1) + \mu(A_2) + \ldots$$

Also, for any n,

$$B_1 = B_n \cup A_1 \cup A_2 \cup \ldots \cup A_{n-1}$$

so

$$\mu(B_1) - \mu(B_n) = \mu(A_1) + \mu(A_2) + \ldots + \mu(A_n)$$

Therefore,

$$\begin{aligned}
\mu(B_1) - \mu(B) &= \mu(A_1) + \mu(A_2) + \ldots \\
&= \lim_{n\to\infty} \mu(A_1) + \mu(A_2) + \ldots + \mu(A_n) \\
&= \lim_{n\to\infty} \mu(B_1) - \mu(B_n) \\
&= \mu(B_1) - \lim_{n\to\infty} \mu(B_n)
\end{aligned}$$

from which we conclude that $\mu(B) = \lim_{n\to\infty} \mu(B_n)$.

4. Define $B_1 = A_1$ and for $j \geq 2$,

$$B_j = A_j \setminus (A_1 \cup \ldots \cup A_{j-1})$$

Then the sets B_j are measurable and disjoint, and $\mu(B_j) \leq \mu(A_j)$ by monotonicity. Therefore, since

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} B_j$$

we obtain by σ-additivity

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(B_j) \leq \sum_{j=1}^{\infty} \mu(A_j)$$

1.4 A Formula for Lebesgue Measure

Suppose $B \subset \bigcup_{j=1}^{\infty} I_j$ (intervals). If B is to be measurable with respect to a measure that extends the length function on intervals, then σ-subadditivity implies $|B| \leq \sum_{j=1}^{\infty} |I_j|$. In particular, $|B|$ is less than or equal to the infimum of all such sums. In Proposition **??** we have seen that the length of an *interval* equals this infimum. Lebesgue measure declares $|B|$ to be equal to this infimum for any Borel set.

Definition 7 *The* Lebesgue measure *of a Borel set B is defined to be*

$$|B| = \inf \left\{ \sum_{j=1}^{\infty} |I_j| \;\middle|\; B \subset \bigcup_{j=1}^{\infty} I_j \text{ (intervals)} \right\}$$

Lebesgue measure is clearly non-negative. The proof that it satisfies σ-additivity is rather involved and we shall be content to simply state this fact without proof.

Theorem 2 *Lebesgue measure is a measure on the Borel sets.*

The key observation of Lebesgue theory is that using countable covers by intervals gives much more precise information about the size of sets than using finite covers.

Examples:

1) Let $B = [0,1] \cap$ be the rational numbers in $[0,1]$. Since $B = \{r_1, r_2, r_3, \ldots\}$ is countable, it is a Borel set. Suppose B is contained in a finite union of intervals, $B \subset I_1 \cup \ldots \cup I_n$. Because B is dense in $[0,1]$, it follows that the union must contain the entire interval, $[0,1] \subset I_1 \cup \ldots \cup I_n$, so by subadditivity, $1 \leq |I_1| + \ldots + |I_n|$. Thus, the best estimate of $|B|$ using *finite* covers by intervals is $|B| \leq 1$.

However, B actually has Lebesgue measure zero and we can show this by estimating $|B|$ using countable covers by intervals. Given $\epsilon > 0$, Let $I_j = (r_j - 2^{-j}\epsilon, r_j + 2^{-j}\epsilon)$. Then $B \subset \bigcup_{j=1}^{\infty} I_j$ and by σ-subadditivity

$$|B| \leq \sum_{j=1}^{\infty} |I_j| = \sum_{j=1}^{\infty} 2^{-j+1}\epsilon = 2\epsilon$$

Since ϵ is arbitrary, we must have that $|B| = 0$. A similar argument can be used to prove:

> *Any countable set of real numbers has Lebesgue measure zero.*

2) Let $A = [0,1] \setminus$ be the irrational numbers and $B = [0,1] \cap$ the rational numbers in $[0,1]$. Then $[0,1] = A \cup B$ is the disjoint union of two Borel sets, so $1 = |A| + |B|$. We know that $|B| = 0$ by example 1), so $|A| = 1$. Thus, even though the rational numbers are dense in $[0,1]$, they are

negligible in size since they have Lebesgue measure zero. All the "weight" of the interval $[0,1]$ is carried by the irrational numbers because they have the same Lebesgue measure as the entire interval.

3) Any open set of real numbers, $A \subset$, is the countable disjoint union of open intervals, $A = \bigcup_{j=1}^{\infty} I_j$, see Theorem **??**. Therefore A is a Borel set and $|A| = \sum_{j=1}^{\infty} |I_j|$. By taking complements, it follows that any closed set of real numbers is a Borel set. The definition of Lebesgue measure also implies that we can "approximate" any Borel set B with an open set in the sense that given any $\epsilon > 0$, there is an countable open cover $A = \bigcup_{j=1}^{\infty} I_j \supset B$ such that

$$|A \setminus B| \leq \epsilon$$

By making the same argument with the complement, $B^c \subset A_0 = \bigcup_{j=1}^{\infty} I_j$, we see that there is also an approximation of B by a closed set $C = A_0^c \subset B$,

$$|B \setminus C| = |A_0 \cap B| = |A_0 \setminus B^c| < \epsilon$$

4) *The Cantor Set.* Let $C_0 = [0,1]$, and let C_1 be the result of removing the segment $(1/3, 2/3)$ from C_0,

$$C_1 = [0, 1/3] \cup [2/3, 1]$$

Let C_2 be the result of removing the middle thirds of the intervals in C_1,

$$C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$$

Continuing in this way we obtain a sequence of compact sets C_n such that

- $C_1 \supset C_2 \supset C_3 \supset \ldots$
- C_n is the union of 2^n intervals, each of length $1/3^n$. In particular, $|C_n| = 2^n/3^n$.

The intersection of these sets is called the *Cantor set*,

$$C = \bigcap_{n=1}^{\infty} C_n$$

It is clearly non-empty and bounded; it is also closed (by Theorem **??**) and therefore compact (by Theorem **??**).

The Cantor set has no point in common with the intervals

$$\Big(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\Big) \quad 0 \leq k < 3^{m-1} - 1$$

since these are the ones that are deleted. Any $(a,b) \subset [0,1]$ contains such an interval (choose m such that $1/3^m < (b-a)/6$), so the Cantor set itself contains no intervals.

The Cantor set can also be described as the set of numbers in $[0,1]$ that have a representation in base-3 using only the numbers 0 and 2. An argument similar to the one that shows $[0,1]$

is uncountable can be used to prove the Cantor set is uncountable: If C were countable, then $C = \{c_1, c_2, \ldots\}$. We construct a number $x \in C$ that is not on the list as follows. The i-th digit of x (base 3) is 0 if the i-th digit of c_i is 2, and the i-th digit of x is 2 if the i-th digit of c_i is 0. The digits for x involve only 0's and 2's, so $x \in C$. Therefore, $x = c_i$ for some i. But, for any i, $x \neq c_i$ because their i-th digits are not equal. This contradiction proves that C is uncountable.

Finally we remark that the Cantor set has Lebesgue measure zero. By Theorem **??**,

$$|C| = \left| \bigcap_{n=1}^{\infty} C_n \right| = \lim_{n\to\infty} |C_n| = \lim_{n\to\infty} \frac{2^n}{3^n} = 0$$

1.5 Other Examples of Measures

1) *Lebesgue Measure on* ^{n}. Lebesgue measure can be easily extended to ^{n} by replacing intervals with rectangles, $R = I_1 \times \cdots \times I_n$, where $I_j \subset$ is an interval. The measure (volume) of R is

$$|R| = |I_1| \cdot |I_2| \cdots |I_n|$$

If denotes the field of finite unions of rectangles, then the sets in the σ-field generated by are again called the Borel sets. If B is a Borel set, then the Lebesgue measure of B is

$$|B| = \inf \left\{ \sum_{j=1}^{\infty} |R_j| \;\middle|\; B \subset \bigcap_{j=1}^{\infty} R_j \text{ (rectangles)} \right\}$$

This extended Lebesgue measure satisfies σ-additivity and so is a measure on the Borel sets in ^{n}.

2) *Counting Measure.* For any subset of natural numbers, $A \subset$, let $\mu(A)$ denote the number of elements in A ($\mu(A) = \infty$ if A has an infinite number of elements). The measure axioms are trivially satisfied, so μ is a measure on the subsets of , called the *counting measure*. We shall see that the integration theory associated with this measure is the theory of convergent series.

3) *Probability Measure.* Let X be a finite set $\{x_1, \ldots, x_n\}$ and let $p_1, \ldots, p_n$ be any non-negative real numbers or ∞. For any subset $A \subset X$, we define

$$\mu(A) = \sum_{x_j \in A} p_j$$

The measure axioms are again trivially verified. In fact, it is not hard to show that *any* measure on X must have this form. If the values p_j also satisfy

$$\mu(X) = \sum_{j=1}^{\infty} p_j = 1$$

then we can interpret them as probabilities: X is the sample space, p_j is the probability that x_j occurs, and $\mu(A)$ is the probability of an event $A \subset X$ (i.e., that an outcome x is in A). In fact, any space X that has a measure defined on a σ-field of subsets of X satisfying $\mu(X) = 1$ can be interpreted as defining probabilities on events (= measurable sets) $A \subset X$.

1.6 Hausdorff Measure and Dimension

Lebesgue measure on satisfies a *scaling property*: if a Borel set B is dilated by a factor of t,

$$tB = \{tx \mid x \in B\}$$

then the Lebesgue measure of tB dilates by the same factor,

$$|tB| = t|B|$$

If B is a Borel set in ^{n}, then the scaling factor is t^n,

$$|tB| = t^n|B|$$

So, for example, if we double the side of a cube, then its volume increases by a factor of $2^3 = 8$. It is possible to construct measures on and on ^{n} that scale by a factor of t^α for any $\alpha \geq 0$ by considering sums of the form $\sum_{j=1}^{\infty} |I_j|^\alpha$ where $B \subset \bigcup_{j=1}^{\infty} I_j$. For technical reasons, the infimum of such sums does not lead to a measure directly. The definition must be modified as follows. A countable collection of intervals $\{I_j\}$ is called an ϵ-*cover* of a set B if $B \subset \bigcup I_j$ and $|I_j| < \epsilon$, $\forall\, j$.

Definition 8 *For any set $B \subset$ define*

$$\mu_\epsilon^\alpha(B) = \inf\left\{ \sum_{j=1}^{\infty} |I_j|^\alpha \;\middle|\; \{I_j\} \text{ is an } \epsilon\text{-cover of } B \right\}$$

The Hausdorff measure of B is defined to be

$$\mu^\alpha(B) = \lim_{\epsilon \to 0} \mu_\epsilon^\alpha(B) = \sup_{\epsilon > 0} \mu_\epsilon^\alpha(B)$$

As ϵ decreases, the collection of ϵ-covers of B is reduced, and hence the infimum, $\mu_\epsilon^\alpha(B)$, increases. Therefore, the limit of $\mu_\epsilon^\alpha(B)$ always exists (possibly ∞) and equals the supremum of $\mu_\epsilon^\alpha(B)$. The Hausdorff measure μ^α is, in fact, a measure on the Borel sets for any $\alpha \geq 0$, although we shall omit the proof. It is easy to see that μ^0 is the counting measure and μ^1 is the usual Lebesgue measure. More generally, Hausdorff measure can be extended to ^{n} by replacing the intervals I_j with closed sets R_j and $|I_j|$ with the diameter of R_j,

$$d(R_j) = \sup\{|x - y| \mid x, y \in R_j\}$$

It can be shown that μ^n is a multiple of Lebesgue measure on the Borel sets in ^{n}. In fact, $\mu^n(B) = c_n|B|$, where c_n is the volume of the n-dimensional ball of diameter 1.

The desired scaling property for Hausdorff measure is clear from the definition,

$$\mu^\alpha(tB) = t^\alpha \mu^\alpha(B)$$

Since we can take the exponent of t as an indication of the dimension of the set, Hausdorff measures with varying α lead, via the following lemma, to the concept of "fractional dimensions."

$$\mu^{\alpha}(B) < \infty \Rightarrow \mu^{\beta}(B) = 0, \forall\, \beta > \alpha$$

$$\mu^{\alpha}(B) > 0 \Rightarrow \mu^{\beta}(B) = \infty, \forall\, \beta < \alpha$$

Let $B \subset \bigcup_{j=1}^{\infty} I_j$ with $I_j \leq \epsilon$. If $\beta > \alpha$, then

$$\sum_{j=1}^{\infty} |I_j|^{\beta} = \sum_{j=1}^{\infty} |I_j|^{\beta-\alpha} |I_j|^{\alpha} \leq \epsilon^{\beta-\alpha} \sum_{j=1}^{\infty} |I_j|^{\alpha}$$

On the other hand, if $\beta < \alpha$, then similarly

$$\sum_{j=1}^{\infty} |I_j|^{\beta} \geq \epsilon^{\beta-\alpha} \sum_{j=1}^{\infty} |I_j|^{\alpha}$$

Therefore, if $\beta > \alpha$,

$$\mu_{\epsilon}^{\beta}(B) \leq \epsilon^{\beta-\alpha} \mu_{\epsilon}^{\alpha}(B) \to 0 \text{ as } \epsilon \to 0$$

so $\mu^{\beta}(B) = 0$; while if $\beta < \alpha$,

$$\mu_{\epsilon}^{\beta}(B) \geq \epsilon^{\beta-\alpha} \mu_{\epsilon}^{\alpha}(B) \to \infty \text{ as } \epsilon \to 0$$

so $\mu^{\beta}(B) = \infty$.

Definition 9 *The* Hausdorff dimension *of a Borel set B is the unique value α such that $\mu^{\beta}(B) = \infty$, $\forall\, \beta < \alpha$ and $\mu^{\beta}(B) = 0$, $\forall\, \beta > \alpha$. A Borel set whose Hausdorff dimension is not an integer is called a* fractal.

Note that if we can find a value of α for which $0 < \mu^{\alpha}(B) < \infty$, then the Hausdorff dimension of B is α.

Example: Let C be the Cantor set (see example 4 on p. 87). We saw that the Lebesgue measure of C is $|C| = 0$. However, the Hausdorff measure of C is $\mu^{\alpha}(C) = 1$ where

$$\alpha = \log(2)/\log(3) \cong 0.6308$$

This implies that Hausdorff dimension of C is α and shows that C is a fractal. To prove $\mu^{\alpha}(C) = 1$ we first show that $\mu^{\alpha}(C) \leq 1$. Recall that

$$C = \bigcap_{n=1}^{\infty} C_n \subset C_n = \bigcup_{i=1}^{2^n} E_n^i$$

for certain disjoint intervals E_n^i with $|E_n^i| = 1/3^n$. Given $\epsilon > 0$, choose n such that $1/3^n < \epsilon$. Then

$$\mu_{\epsilon}^{\alpha}(C) \leq \sum_{i=1}^{2^n} |E_n^i|^{\alpha} = \sum_{j=1}^{2^n} (\frac{1}{3^n})^{\alpha} = 2^n (\frac{1}{3^n})^{\alpha} = (\frac{2}{3^{\alpha}})^n = 1$$

since $3^\alpha = 2$. So $\mu^\alpha(C) = \lim_{\epsilon\to 0}\mu^\alpha_\epsilon(C) \le 1$.

It is much harder to prove the reverse inequality, $\mu^\alpha(C) \ge 1$. We need the following inequality.

Let F be an interval with $|F| = 1/3^n$ and let F_L, F_R be adjacent intervals on the left and right of F, respectively, with $|F_L|, |F_R| \le 1/3^n$. Let $U = F_L \cup F \cup F_R$ and let $\alpha = \log(2)/\log(3)$. Then

$$|F_L|^\alpha + |F_R|^\alpha \le |U|^\alpha$$

Let $|F_L| = 1/3^n - x$ and $|F_R| = 1/3^n - y$ for some $0 \le x, y \le 1/3^n$. Then $|U| = 1/3^{n-1} - x - y$ and we must prove that the function

$$f(x,y) = \Big(\frac{1}{3^{n-1}} - x - y\Big)^\alpha - \Big(\frac{1}{3^n} - x\Big)^\alpha - \Big(\frac{1}{3^n} - y\Big)^\alpha \ge 0$$

on the domain $0 \le x, y \le 1/3^n$. Note that

$$f(0,0) = \Big(\frac{1}{3^{n-1}}\Big)^\alpha - 2\Big(\frac{1}{3^n}\Big)^\alpha = \frac{1}{2^{n-1}} - 2\frac{1}{2^n} = 0$$

since $3^\alpha = 2$. Also,

$$f_x(x,y) = -\alpha\Big(\frac{1}{3^{n-1}} - x - y\Big)^{\alpha-1} + \alpha\Big(\frac{1}{3^n} - x\Big)^{\alpha-1} = 0$$

only for $y = 2/3^n$ which lies outside the domain. Since

$$f_x(0,0) = -\alpha\Big(\frac{1}{3^{n-1}}\Big)^{\alpha-1} + \alpha\Big(\frac{1}{3^n}\Big)^{\alpha-1} = \alpha\Big(\frac{3^n}{2^n} - \frac{3^{n-1}}{2^{n-1}}\Big) > 0$$

it follows that $f_x(x,y) > 0$ on the domain. By symmetry, the same holds for $f_y(x,y)$. Therefore, $f(0,0) = 0$ is the minimum of f on the domain and hence $f(x,y) \ge 0$ as claimed.

Now suppose $C \subset \bigcup_{j=1}^\infty I_j$ is an ϵ-cover of C. Since C is compact, $\exists\, N$ such that $C \subset \bigcup_{j=1}^N I_j$. Each interval E_n^i of C_n is adjacent to exactly one open interval G_n^i of length $1/3^n$ (a middle third that was removed) and another interval $E_n^{i\pm1}$ on the other side of G_n^i such that

$$E_n^i \cup G_n^i \cup E_n^{i\pm1} = E_{n-1}^k$$

for some k. By the lemma, we have for each $1 \le j \le N$,

$$|E_n^i \cap I_j|^\alpha + |E_n^{i\pm1} \cap I_j|^\alpha \le |E_{n-1}^k \cap I_j|^\alpha$$

Repeating this construction recursively with the intervals E_m^i in C_m for $0 \le m \le n$, we find that

$$\sum_{i=1}^{2^n} |E_n^i \cap I_j|^\alpha \le \sum_{i=1}^{2^{n-1}} |E_{n-1}^i \cap I_j|^\alpha \le \ldots \le |E_0 \cap I_j|^\alpha = |I_j|^\alpha$$

Now $|E_n^i \cap I_j| \ne 0$ only if $E_n^i \cap I_j \ne \emptyset$ and in this case $|E_n^i \cap I_j| = |E_n^i|$ (i.e., $E_n^i \subset I_j$) with at most two exceptions—those E_n^i that overlap the endpoints of the interval I_j. Therefore,

$$\sum_{E_n^i \cap I_j \ne \emptyset} |E_n^i|^\alpha \;-\; \frac{2}{3^n} \;\le\; |I_j|^\alpha$$

Since the intervals I_j, $1 \le j \le N$, cover C, together they must intersect all the sets E_n^i. So, adding the above inequalities for $1 \le j \le N$, gives

$$\sum_{i=1}^{2^n} |E_n^i|^\alpha - \frac{2N}{3^n} \;=\; \sum_{j=1}^{N} \Big(\sum_{E_n^i \cap I_j \neq \emptyset} |E_n^i|^\alpha - \frac{2}{3^n} \Big) \;\le\; \sum_{j=1}^{N} |I_j|^\alpha$$

Since $\sum_{i=1}^{2^n} |E_n^i|^\alpha = 1$ (see above), we find

$$1 - \frac{2N}{3^n} \le \mu_\epsilon^\alpha(C) \le \mu^\alpha(C)$$

for all n. Therefore, $1 \le \mu^\alpha(C)$, as claimed.

For sets that are self-symmetric, like the Cantor set, there are often simple heuristic arguments to determine their dimension using the scaling property. If we scale a set by a factor of t and the resulting set gives of k copies of the original set, then the scaling property implies $t^\alpha = k$ and so its dimension should be $\alpha = \log(k)/\log(t)$. For example, if we multiply the Cantor set by three, we get two copies of the original Cantor set (one on the interval $[0, 1]$, and the other on the interval $[2, 3]$). Therefore, its dimension is $\alpha = \log(2)/\log(3)$. This argument is heuristic because we do not know *a priori* that there is a way to measure the size of the Cantor set. That is, we need a function that satisfies the scaling property $\mu(tC) = t^\alpha \mu(C)$ with $\mu(C) \neq 0$ so that we can argue that $\mu(3C) = 3^\alpha \mu(C) = 2\mu(C) \Rightarrow 3^\alpha = 2$.