1 Lebesgue Integration

1.1 Measurable Functions

Definition 1 *Let X be a set with a measure μ defined on a σ-field of subsets . Then $(X, , \mu)$ is called a* measure space *and the sets in are called* measurable sets.

We are primarily interested in the case where $X =$ (or ^{n}), is the σ-field of Borel sets, and μ is Lebesgue measure. Nevertheless, we shall assume throughout this section that $(X, , \mu)$ is a general measure space because the integration theory we want to discuss would not become simpler if we restricted ourselves to Lebesgue measure on . In fact, the essential features of the theory are easier to grasp when it is seen that they depend only on the σ-additivity of the measure μ on the σ-field .

Definition 2 *A function $f : X \to$ is* measurable *if $f^{-1}(B)$ is measurable for all Borel sets $B \subset$.*

Remarks:

1) Since intervals generate the Borel sets and f^{-1} preserves set operations, it is clear that a function f is measurable if and only if $f^{-1}(I)$ is measurable for any *interval* I (open or closed). In fact, it is enough to check that $f^{-1}(a, \infty)$ is measurable $\forall\, a \in$, since

$$\begin{aligned}
(-\infty, a] &= \setminus(a, \infty) \\
(-\infty, a) &= \bigcap_{n=1}^{\infty}(-\infty, a + 1/n] \\
[a, \infty) &= \setminus(-\infty, a) \\
(a, b) &= (-\infty, b) \cap (a, \infty)
\end{aligned}$$

2) If f is continuous, then for any open interval I, $f^{-1}(I)$ is open and hence measurable. Therefore, f is measurable.

3) If $f : X \to D \subset$ is measurable and $g : D \to$ is measurable (with respect to Lebesgue measure on Borel sets), then the composition $g \circ f$ is again measurable: If B is a Borel set, then $g^{-1}(B)$ is a Borel set and so $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$ is measurable. It is worth mentioning that there is a slightly larger σ-field of subsets of , called the *Lebesgue sets*, that contains the Borel sets and to which Lebesgue measure can be extended. The Lebesgue sets are taken to be the standard σ-field of measurable sets on by many mathematicians. However, if we assume that f and g are measurable with respect to Lebesgue sets, then the composition $g \circ f$ need *not* be measurable.

Theorem 1 *Let $\{f_n\}$ be a sequence of measurable functions. Then the functions*

$$g(x) = \sup\{f_n(x)\}$$

$$\begin{aligned} h(x) &= \inf\{f_n(x)\} \\ j(x) &= \limsup\{f_n(x)\} \\ k(x) &= \liminf\{f_n(x)\} \end{aligned}$$

are measurable.

First note that for any $a \in$, $g(x) > a \Leftrightarrow f_n(x) > a$ for some n. Therefore,

$$g^{-1}(a,\infty) = \bigcup_{n=1}^{\infty} f_n^{-1}(a,\infty)$$

The sets $f_n^{-1}(a,\infty)$ are measurable by assumption, so $g^{-1}(a,\infty)$ is measurable. By Remark 1, g is measurable. A similar proof shows h is measurable.

By what we have just proved,

$$g_k(x) = \sup\{f_n(x) \mid n \geq k\}$$

is measurable and hence

$$j(x) = \limsup\{f_n\} = \lim_{k\to\infty}\{g_k(x)\} = \inf\{g_k(x)\}$$

is measurable. The proof for $\liminf$ is similar.

Corollary 2 *(a) If f and g are measurable, then $\max\{f,g\}$ and $\min\{f,g\}$ are measurable. In particular,*

$$f^+ = \max\{f,0\} \quad \textit{and} \quad f^- = -\min\{f,0\}$$

are measurable.

(b) The limit of a convergent sequence of measurable functions is measurable.

Theorem 3 *Let $f,g : X \to$ be measurable functions and let $F :^2 \to$ be continuous. Then $h(x) = F(f(x),g(x))$ is measurable. In particular, $f+g$, $f-g$, fg, and f/g $(g \neq 0)$ are measurable.*

For any $a \in$, $F^{-1}(a,\infty)$ is open, and hence can be written as a countable union of open rectangles,

$$F^{-1}(a,\infty) = \bigcup_{n=1}^{\infty} R_n$$

where $R_n = (a_n,b_n) \times (c_n,d_n)$. Now,

$$\begin{aligned} x \in h^{-1}(a,\infty) &\Leftrightarrow (f(x),g(x)) \in F^{-1}(a,\infty) \\ &\Leftrightarrow (f(x),g(x)) \in R_n \text{ for some } n \\ &\Leftrightarrow x \in f^{-1}(a_n,b_n) \cap g^{-1}(c_n,d_n) \text{ for some } n \end{aligned}$$

Therefore,

$$h^{-1}(a,\infty) = \bigcup_{n=1}^{\infty} f^{-1}(a_n, b_n) \cap g^{-1}(c_n, d_n)$$

Since $f^{-1}(a_n, b_n)$ and $g^{-1}(c_n, d_n)$ are measurable, so is $h^{-1}(a,\infty)$, and hence h is measurable.

Summing up, we may say that all ordinary operations of analysis, including limit operations, when applied to measurable functions, lead to measurable functions.

1.2 Simple Functions

Definition 3 *A function $s : X \to$ is called* simple *if the range of s is finite. For any subset $E \subset X$, the* characteristic function of E *is defined to be*

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

Suppose the range of a simple function s consists of the distinct numbers $c_1, \ldots, c_n$. Let $E_i = s^{-1}(c_i)$. Then

$$s = \sum_{i=1}^{n} c_i \chi_{E_i}$$

that is, every simple function is a finite linear combination of characteristic functions. It is clear that s is measurable if and only if the sets E_i are measurable.

Simple functions are more useful than they appear at first sight.

Theorem 4 (Approximation Theorem) *Any function $f : X \to$ can be approximated by simple functions, that is, there is a sequence of simple functions, $\{s_n\}$, such that $s_n \to f$ pointwise. If f is measurable, then the s_n may be chosen to be measurable. If $f \geq 0$, then the s_n may be chosen to be monotonically increasing. If f is bounded, then the s_n converge uniformly to f.*

Suppose $f \geq 0$. For $n \in$ and $i = 1, 2, 3, \ldots, n2^n$, define

$$E_i^n = f^{-1}[(i-1)/2^n, i/2^n)$$

and $E_\infty^n = f^{-1}[n, \infty)$. Let

$$s_n = \sum_{i=1}^{n2^n} \frac{(i-1)}{2^n} \chi_{E_i^n} + n\chi_{E_\infty^n}$$

We now show that for any $x \in X$, $s_n(x) \to f(x)$. Given $\epsilon > 0$, choose an integer N such that $f(x) < N$ and $1/2^N < \epsilon$. For any $n \geq N$, there exists a positive integer $i \leq n2^n$ such that $f(x) \in [(i-1)/2^n, i/2^n)$. Since $s_n(x) = (i-1)/2^n$, we have

$$|f(x) - s_n(x)| < \frac{1}{2^n} < \epsilon$$

If f is bounded then N can be chosen independent of x and the above inequality holds $\forall\, n \geq N$ and $\forall\, x \in X$, hence the convergence is uniform.

To see that the s_n are monotonically increasing in n, note that

$$\Big[\frac{i-1}{2^n}, \frac{i}{2^n}\Big) = \Big[\frac{2i-2}{2^{n+1}}, \frac{2i-1}{2^{n+1}}\Big) \cup \Big[\frac{2i-1}{2^{n+1}}, \frac{2i}{2^{n+1}}\Big)$$

Thus,

$$f(x) \in \Big[\frac{2i-2}{2^{n+1}}, \frac{2i-1}{2^{n+1}}\Big) \Rightarrow s_n(x) = \frac{i-1}{2^n} = s_{n+1}(x)$$

while

$$f(x) \in \Big[\frac{2i-1}{2^{n+1}}, \frac{2i}{2^{n+1}}\Big) \Rightarrow s_n(x) = \frac{i-1}{2^n} < \frac{2i-1}{2^{n+1}} = s_{n+1}(x)$$

Similarly,

$$f(x) \in [n, \infty) \Rightarrow s_n(x) = n \leq \frac{n2^{n+1}+k}{2^{n+1}} = s_{n+1}(x)$$

for some $0 \leq k \leq 2^{n+1}$.

For a general function, write $f = f^+ - f^-$ and apply the preceding construction to f^+ and f^-. Finally, if f is measurable, then the sets E_i^n are measurable, and hence the simple functions s_n are measurable.

1.3 The Lebesgue Integral

We now define the Lebesgue integral on a measure space $(X, , \mu)$. The definition is built up in stages, similar to the process used for the Riemann integral. First we define the Lebesgue integral of a measurable simple function. The definition is straightforward and similar to the definition of the integral of a step function in the Riemann theory. Then, since a non-negative measurable function f can be approximated by a monotonically increasing sequence of measurable simple functions, we define the integral of f to be the supremum of the integrals of measurable simple functions $\leq f$. Finally, an arbitrary measurable function can be split $f = f^+ - f^-$ where f^+ and f^- are non-negative, and the integral is the corresponding combination of the integrals of f^+ and f^-.

Definition 4 *The* Lebesgue integral *of a measurable function over a measurable set A is defined as follows:*

1. *If $s = \sum_{i=1}^{n} c_i \chi_{E_i}$ is a simple measurable function then*

$$\int_A s\, d\mu = \sum_{i=1}^{n} c_i \mu(A \cap E_i)$$

2. *If f is a non-negative measurable function, then*

$$\int_A f\, d\mu = \sup\Big\{ \int_A s\, d\mu \mid 0 \leq s \text{ (simple) } \leq f \Big\}$$

3. *If f is an arbitrary measurable function, then consider*

$$\int_A f^+ \, d\mu, \quad \int_A f^- \, d\mu$$

where $f^+ = \max\{f, 0\}$ *and* $f^- = \max\{-f, 0\}$ *are non-negative measurable functions and* $f = f^+ - f^-$ *(see Corollary* **??***). If at least one of these integrals is finite, we define*

$$\int_A f \, d\mu = \int_A f^+ \, d\mu - \int_A f^- \, d\mu$$

If both integrals are finite then we say that f is Lebesgue integrable *on A with respect to μ and write $f \in_\mu (A)$.*

Remarks:

1) It should be noted that an integral may have the value $\pm\infty$, but the term *integrable* is only applied when the integral is finite.

2) If s is a simple measurable function, then the value of $\int_A s \, d\mu$ given in 2 and 3 agrees with that given in 1.

3) If $A \subset$ (or ^{n}) and μ is Lebesgue measure, it is customary to drop the subscript μ and write (A) for the Lebesgue measurable functions on A.

Theorem 5 (Basic Properties) *In the following statements, functions and sets are assumed to be measurable.*

1. *If f is bounded on A and $\mu(A) < \infty$, then $f \in_\mu (A)$. In fact, if $m \leq f(x) \leq M$, $\forall\, x \in A$, then*

$$m\mu(A) \leq \int_A f \, d\mu \leq M\mu(A)$$

2. *If $f, g \in_\mu (A)$ and $f(x) \leq g(x)$, $\forall\, x \in A$, then*

$$\int_A f \, d\mu \leq \int_A g \, d\mu$$

3. *If $f \in_\mu (A)$ then $cf \in_\mu (A)$, $\forall\, c \in$, and*

$$\int_A cf \, d\mu = c \int_A f \, d\mu$$

4. *If $\mu(A) = 0$ and $f \in_\mu (A)$, then*

$$\int_A f \, d\mu = 0$$

5. *If f is non-negative and $B \subset A$ then*

$$\int_B f\,d\mu \leq \int_A f\,d\mu$$

6. *If $f \in_\mu (A)$ and $B \subset A$, then $f \in_\mu (B)$.*

These statements are direct consequences of the definitions and their verification is left as an exercise for the reader.

1.4 Further Properties of the Lebesgue Integral

Theorem 6 *(a) If f is measurable and non-negative then for any countable union of disjoint measurable sets, $A = \bigcup_{n=1}^{\infty} A_n$,*

$$\int_A f\,d\mu = \sum_{n=1}^{\infty} \int_{A_n} f\,d\mu$$

(b) The same conclusion holds for $f \in_\mu (A)$.

Statement *(b)* follows immediately from *(a)* by writing $f = f^+ - f^-$ and applying *(a)* to f^+ and f^-.

To prove *(a)*, we first observe that if s is a non-negative simple function,

$$s = \sum_{i=1}^{n} c_i \chi_{E_i}$$

then, by the σ-additivity of μ,

$$\int_A s\,d\mu = \sum_{i=1}^{n} c_i \mu(E_i \cap A) = \sum_{i=1}^{n} \sum_{n=1}^{\infty} c_i \mu(E_i \cap A_n) = \sum_{n=1}^{\infty} \int_{A_n} s\,d\mu$$

(We can rearrange the series because all terms are ≥ 0; they either sum to ∞ or are absolutely convergent.) Now suppose s is a simple function such that $0 \leq s \leq f$. Since $\int_{A_n} s\,d\mu \leq \int_{A_n} f\,d\mu$,

$$\int_A s\,d\mu = \sum_{n=1}^{\infty} \int_{A_n} s\,d\mu \leq \sum_{n=1}^{\infty} \int_{A_n} f\,d\mu$$

Taking the supremum over all simple functions $0 \leq s \leq f$ we obtain

$$\int_A f\,d\mu \leq \sum_{n=1}^{\infty} \int_{A_n} f\,d\mu$$

To show the reverse inequality, let $\epsilon > 0$ and choose a simple function s such that $0 \leq s \leq f$ and such that

$$\int_{A_1} s\, d\mu \geq \int_{A_1} f\, d\mu - \epsilon, \quad \int_{A_2} s\, d\mu \geq \int_{A_2} f\, d\mu - \epsilon$$

Then

$$\int_{A_1 \cup A_2} f\, d\mu \geq \int_{A_1 \cup A_2} s\, d\mu = \int_{A_1} s\, d\mu + \int_{A_2} s\, d\mu \geq \int_{A_1} f\, d\mu + \int_{A_2} f\, d\mu - 2\epsilon$$

Since ϵ is arbitrary, we conclude

$$\int_{A_1 \cup A_2} f\, d\mu \geq \int_{A_1} f\, d\mu + \int_{A_2} f\, d\mu$$

We can repeat this argument a finite number of times to get

$$\int_{A_1 \cup \ldots \cup A_n} f\, d\mu \geq \int_{A_1} f\, d\mu + \ldots + \int_{A_n} f\, d\mu$$

Since $A \supset A_1 \cup \ldots \cup A_n$, we know by Theorem **??**(5) that

$$\int_{A} f\, d\mu \geq \int_{A_1 \cup \ldots \cup A_n} f\, d\mu$$

and so

$$\int_{A} f\, d\mu \geq \int_{A_1} f\, d\mu + \ldots + \int_{A_n} f\, d\mu$$

Since this holds for any n, we conclude

$$\int_{A} f\, d\mu \geq \sum_{n=1}^{\infty} \int_{A_n} f\, d\mu$$

Corollary 7 *Let B and C be disjoint measurable sets and let $A = B \cup C$. If $\mu(C) = 0$, then for any measurable function f,*

$$\int_{A} f\, d\mu = \int_{B} f\, d\mu$$

$$\int_{A} f\, d\mu = \int_{B} f\, d\mu + \int_{C} f\, d\mu = \int_{B} f\, d\mu$$

by Theorem **??** and Theorem **??**(4).

This corollary shows that sets of measure zero can be ignored in integration. A common application is the following. Let f and g be measurable functions defined on a measurable set A, and let

$$B = \{x \in A \mid f(x) = g(x)\}, \quad C = A \setminus B$$

If $\mu(C) = 0$, then

$$\int_{A} f\, d\mu = \int_{B} f\, d\mu = \int_{B} g\, d\mu = \int_{A} g\, d\mu$$

Thus, functions that differ only on a set of measure zero have the same Lebesgue integral.

Definition 5 *We say that a property $P(x)$ holds* almost everywhere *if $P(x)$ holds for every x except for x in a set of measure zero (i.e., $P(x)$ holds for $x \in A \setminus Z$ with $\mu(Z) = 0$).*

We can formulate the previous remark as:

Corollary 8 *Let f be g are measurable functions on a measurable set A. If $f = g$ almost everywhere on A, then*

$$\int_A f \, d\mu = \int_A g \, d\mu$$

Example: The Dirichlet function, $f = \chi_{[0,1]\cap}$, equals the zero function, $g = 0$, almost everywhere on $[0, 1]$, so $\int_{[0,1]} f \, d\mu = \int_{[0,1]} g \, d\mu = 0$.

Theorem 9 *If $f \in_\mu (A)$, then $|f| \in_\mu (A)$ and*

$$\Big| \int_A f \, d\mu \Big| \leq \int_A |f| \, d\mu$$

Let

$$\begin{aligned} B &= \{x \in A \mid f(x) \geq 0\} \\ C &= \{x \in A \mid f(x) < 0\} \end{aligned}$$

so that $A = B \cup C$ is a disjoint union of measurable sets. By Theorem **??**,

$$\int_A |f| \, d\mu = \int_B |f| \, d\mu + \int_C |f| \, d\mu = \int_B f^+ \, d\mu + \int_C f^- \, d\mu < \infty$$

which shows that $|f| \in_\mu (A)$. Since $f \leq |f|$ and $-f \leq |f|$,

$$\int_A f \, d\mu \leq \int_A |f| \, d\mu, \quad -\int_A f \, d\mu \leq \int_A |f| \, d\mu$$

which implies

$$\Big| \int_A f \, d\mu \Big| \leq \int_A |f| \, d\mu$$

Theorem 10 *Suppose f is measurable and $|f| \leq g$ for some $g \in_\mu (A)$. Then $f \in_\mu (A)$.*

$|f| \leq g$ implies $0 \leq f^\pm \leq g$. Therefore $\int_A f^\pm \, d\mu < \infty$ and so $f \in_\mu (A)$.

Theorem 11 (Lebesgue's Monotone Convergence Theorem)
Suppose $\{f_n\}$ is a monotonically increasing sequence of non-negative measurable functions on a measurable set A,

$$0 \leq f_1(x) \leq f_2(x) \leq \dots, \quad \forall x \in A$$

Then,

$$\lim_{n\to\infty}\int_A f_n\,d\mu = \int_A \lim_{n\to\infty} f_n\,d\mu$$

Because the sequences $f_n(x)$ and $\int_A f_n\,d\mu$ are monotonically increasing, their limits exist in $\cup\{\infty\}$,

$$f(x) = \lim_{n\to\infty} f_n(x), \quad I = \lim_{n\to\infty}\int_A f_n\,d\mu$$

Now $f_n(x) \le f(x)$ on A implies $\int_A f_n\,d\mu \le \int_A f\,d\mu$ for each n, and hence the inequality is preserved in the limit,

$$I \le \int_A f\,d\mu$$

We now prove the reverse inequality. Pick c such that $0 < c < 1$, and let s be a simple measurable function such that $0 \le s \le f$. For $n \in$ define

$$A_n = \{x \in A \mid f_n(x) \ge cs(x)\}$$

The assumption $f_1(x) \le f_2(x) \le \ldots$ implies $A_1 \subset A_2 \subset \ldots$ Moreover,

$$A = \bigcup_{n=1}^{\infty} A_n$$

The union is obviously contained in A. Conversely, A is contained in the union since for any $x \in A$ there is an n such that $f_n(x) > cf(x)$ ($f_n(x)$ is monotonically increasing to $f(x)$ and $f(x) > cf(x)$), so $x \in A_n$. Using the fact that $A \supset A_n$ and $f_n(x) \ge cs(x)$ on A_n we get

$$\int_A f_n\,d\mu \ge \int_{A_n} f_n\,d\mu \ge c\int_{A_n} s\,d\mu \tag{1}$$

Let $s = \sum_{i=1}^{k} c_i\chi_{E_i}$ be the description of s in terms of characteristic functions. Since

$$E_i \cap A = \bigcup_{n=1}^{\infty} E_i \cap A_n$$

Theorem **??** implies that

$$\mu(E_i \cap A) = \lim_{n\to\infty}\mu(E_i \cap A_n)$$

Therefore, as $n \to \infty$,

$$\int_{A_n} s\,d\mu = \sum_{i=1}^{\infty} c_i\mu(E_i \cap A_n) \to \sum_{i=1}^{\infty} c_i\mu(E_i \cap A) = \int_A s\,d\mu$$

and (**??**) becomes in the limit as $n \to \infty$,

$$I = \lim_{n\to\infty}\int_A f_n\,d\mu \ge c\int_A s\,d\mu$$

Letting $c \to 1$ we obtain

$$I \ge \int_A s\,d\mu$$

Now taking the supremum over all simple functions $0 \le s \le f$ gives

$$I \ge \int_A f \, d\mu$$

Lebesgue's Monotone Convergence Theorem in conjunction with the Approximation Theorem **??** is a powerful tool for extending properties of integrals from simple functions to more general functions. The non-negativity assumption can often be worked around as the proof of the following theorem demonstrates.

Theorem 12 *If $f_1, f_2 \in_\mu (A)$ then $f_1 + f_2 \in_\mu (A)$ and*

$$\int_A f_1 + f_2 \, d\mu = \int_A f_1 \, d\mu + \int_A f_2 \, d\mu$$

First suppose $f_1 \ge 0$ and $f_2 \ge 0$. If these functions are simple, say $f_k = \sum c_{ki} \chi_{E_{ki}}$, $k = 1, 2$, then

$$f_1 + f_2 = \sum_{i=1}^{n_1} c_{1i} \chi_{E_{1i}} + \sum_{j=1}^{n_2} c_{2j} \chi_{E_{2j}}$$

and by definition

$$\begin{aligned} \int_A f_1 + f_2 \, d\mu &= \sum_{i=1}^{n_1} c_{1i} \mu(A \cap E_{1i}) + \sum_{j=1}^{n_2} c_{2j} \mu(A \cap E_{2j}) \\ &= \int_A f_1 + \int_A f_2 \, d\mu \end{aligned}$$

If the f_k are not simple, then by Theorem **??** there is a sequence of monotonically increasing simple functions $s_{kn} \to f_k$, $k = 1, 2$, with $s_{1n} + s_{2n}$ monotonically increasing to $f_1 + f_2$. Lebesgue's Monotone Convergence Theorem implies

$$\lim_{n \to \infty} \int_A s_{1n} + s_{2n} \, d\mu = \int_A f_1 + f_2 \, d\mu$$

Linearity for simple functions gives

$$\begin{aligned} \lim_{n \to \infty} \int_A s_{1n} + s_{2n} \, d\mu &= \lim_{n \to \infty} \int_A s_{1n} \, d\mu + \int_A s_{2n} \, d\mu \\ &= \int_A f_1 \, d\mu + \int_A f_2 \, d\mu \end{aligned}$$

Therefore,

$$\int_A f_1 + f_2 \, d\mu = \int_A f_1 \, d\mu + \int_A f_2 \, d\mu$$

Next suppose $f_1 \ge 0$ and $f_2 \le 0$. Let $f = f_1 + f_2$ and define

$$A^+ = \{x \in A \mid f(x) \ge 0\}, \quad A^- = \{x \in A \mid f(x) < 0\}$$

Then f, f_1, and $-f_2$ are non-negative on A^+. Applying what we have just proved to $f_1 = f+(-f_2)$,

$$\int_{A^+} f_1\,d\mu = \int_{A^+} f\,d\mu + \int_{A^+} (-f_2)\,d\mu = \int_{A^+} f\,d\mu - \int_{A^+} f_2\,d\mu$$

or

$$\int_{A^+} f\,d\mu = \int_{A^+} f_1\,d\mu + \int_{A^+} f_2\,d\mu \tag{2}$$

Similarly, $-f$, f_1 and $-f_2$ are non-negative on A^- and $(-f_2) = (-f) + f_1$ so

$$\int_{A^-} (-f_2)\,d\mu = \int_{A^-} (-f)\,d\mu + \int_{A^-} f_1\,d\mu$$

or

$$\int_{A^-} f\,d\mu = \int_{A^-} f_1\,d\mu + \int_{A^-} f_2\,d\mu \tag{3}$$

Adding (**??**) and (**??**), and using the fact that A^+, A^- are measurable disjoint sets with $A = A^+ \cup A^-$, we get by Theorem **??**,

$$\begin{aligned}
\int_A f\,d\mu &= \int_{A^+} f\,d\mu + \int_{A^-} f\,d\mu \\
&= \int_{A^+} f_1\,d\mu + \int_{A^+} f_2\,d\mu + \int_{A^-} f_1\,d\mu + \int_{A^-} f_2\,d\mu \\
&= \int_A f_1\,d\mu + \int_A f_2\,d\mu
\end{aligned}$$

In the general case, A can be decomposed into four sets A_i on each of which f_1 and f_2 have a constant sign. The two cases we have proved so far imply

$$\int_{A_i} f_1 + f_2\,d\mu = \int_{A_i} f_1\,d\mu + \int_{A_i} f_2\,d\mu \quad (i = 1, 2, 3, 4)$$

By adding these four equations as before we get the desired formula.

Lebesgue's Monotone Convergence Theorem can also be applied to integrals of series. The corresponding theorem for Riemann integrals requires much stronger assumptions.

Theorem 13 *Suppose f_n is a sequence of non-negative measurable functions on a measurable set A. Then*

$$\int_A \sum_{n=1}^{\infty} f_n\,d\mu = \sum_{n=1}^{\infty} \int_A f_n\,d\mu$$

The partial sums, $s_k = \sum_{n=1}^{k} f_n$, form a monotonically increasing sequence of measurable functions converging to

$$\sum_{n=1}^{\infty} f_n = \lim_{k\to\infty} s_k$$

By Lebesgue's Monotone Convergence Theorem and Theorem **??**,

$$\begin{aligned}
\int_A \sum_{n=1}^{\infty} f_n \, d\mu &= \int_A \lim_{k\to\infty} s_k \, d\mu = \lim_{k\to\infty} \int_A s_k \, d\mu \\
&= \lim_{k\to\infty} \int_A \sum_{n=1}^{k} f_n \, d\mu = \lim_{k\to\infty} \sum_{n=1}^{k} \int_A f_n \, d\mu \\
&= \sum_{n=1}^{\infty} \int_A f_n \, d\mu
\end{aligned}$$

[Fatou's Lemma] Suppose f_n is a sequence of non-negative measurable functions on a measurable set A. Then

$$\int_A \liminf f_n \, d\mu \le \liminf \int_A f_n \, d\mu$$

Define

$$g_k(x) = \inf\{f_n(x) \mid n \ge k\}$$

Then g_k is non-negative, measurable, monotonically increasing to

$$\liminf f_n(x) = \lim_{k\to\infty} g_k(x)$$

Moreover,

$$\int_A g_k \, d\mu \le \int_A f_n \, d\mu, \quad \forall\, n \ge k$$

so

$$\int_A g_k \, d\mu \le \inf \Big\{ \int_A f_n \, d\mu \mid n \ge k \Big\}$$

Therefore, by Lebesgue's Monotone Convergence Theorem,

$$\begin{aligned}
\int_A \liminf f_n \, d\mu &= \int_A \lim_{k\to\infty} g_k \, d\mu = \lim_{k\to\infty} \int_A g_k \, d\mu \\
&\le \lim_{k\to\infty} \inf \Big\{ \int_A f_n \, d\mu \mid n \ge k \Big\} \\
&= \liminf \int_A f_n \, d\mu
\end{aligned}$$

It is easy to find examples that show strict inequality may hold in Fatou's lemma. Let

$$g(x) = \begin{cases} 0, & 0 \le x \le 1/2 \\ 1, & 1/2 < x \le 1 \end{cases}$$

and define $f_n = g(x)$ if n is odd and $f_n = g(1-x)$ if n is even. Then

$$\liminf f_n(x) = 0, \quad \forall x \in [0,1]$$

but

$$\int_0^1 f_n(x) dx = 1/2$$

Theorem 14 (Lebesgue's Dominated Convergence Theorem)
Suppose f_n is a sequence of measurable functions on a measurable set A that converges pointwise on A,

$$f(x) = \lim_{n\to\infty} f_n(x), quad \forall\, x \in A$$

Assume there is a function $g \in_\mu (A)$ such that $\forall\, n$,

$$|f_n(x)| \leq g(x), \quad \forall\, x \in A$$

Then

$$\lim_{n\to\infty} \int_A f_n \, d\mu = \int_A \lim_{n\to\infty} f_n \, d\mu$$

We first observe that $f_n \in_\mu (A)$ and $f \in_\mu (A)$ by Theorem **??**. Since $f_n + g \geq 0$, we may apply Fatou's Lemma to get

$$\int_A (f+g)\, d\mu \leq \liminf \int_A (f_n + g)\, d\mu$$

or

$$\int_A f \, d\mu \leq \liminf \int_A f_n \, d\mu$$

Similarly, since $g - f_n \geq 0$, we get

$$\int_A (g-f)\, d\mu \leq \liminf \int_A (g - f_n)\, d\mu$$

so

$$-\int_A f \, d\mu \leq \liminf - \int_A f_n \, d\mu$$

which is equivalent to

$$\int_A f \, d\mu \geq \limsup \int_A f_n \, d\mu$$

Therefore,

$$\int_A f \, d\mu \leq \liminf \int_A f_n \, d\mu \leq \limsup \int_A f_n \, d\mu \leq \int_A f \, d\mu$$

so the $\limsup$ equals the $\liminf$, and it follows that the limit exists,

$$\int_A f \, d\mu = \lim_{n\to\infty} \int_A f_n \, d\mu$$

Corollary 15 *If $\mu(A) < \infty$, $\{f_n\}$ is uniformly bounded on A, and $f(x) = \lim_{n\to\infty} f_n(x)$, $\forall\, x \in A$, then*

$$\lim_{n\to\infty} \int_A f_n \, d\mu = \int_A f \, d\mu$$

A converse to Corollary **??** also holds.

Theorem 16 *Let $f, g \in_\mu (A)$. Suppose $f \geq g$ on A and*

$$\int_A f\, d\mu = \int_A g\, d\mu$$

Then $f = g$ almost everywhere on A.

Let

$$B_n = \{x \in A \mid f(x) - g(x) \geq 1/n\}$$

Then $B_1 \subset B_2 \subset B_3 \subset \dots$ and

$$B = \bigcup_{n=1}^{\infty} B_n = \{x \in A \mid f(x) - g(x) > 0\}$$

By our assumption and linearity (Theorem **??**), $\int_A f - g\, d\mu = 0$, so

$$0 = \int_{B_n} f - g\, d\mu \geq \frac{1}{n}\mu(B_n)$$

Hence, $\mu(B_n) = 0$, $\forall\, n$. By Theorem **??**(2),

$$\mu(B) = \lim_{n \to \infty} \mu(B_n) = 0$$

and therefore $f = g$ except on a set B of measure 0.

1.5 Comparison with the Riemann Integral

We now show that the Lebesgue integral gives the same value as the Riemann integral on an interval. We shall also show that Riemann integrable functions are subject to the rather stringent condition of being continuous almost everywhere. One of the greatest advantages of the Lebesgue integral, aside from the fact that it provides a larger class of functions that can be integrated, is the ease with which limit operations can be handled as demonstrated by the Lebesgue Convergence Theorems **??**, **??**. Limits of Riemann integrable functions need not be Riemann integrable; in the Lebesgue theory this difficulty is almost eliminated since limits of a measurable functions are always measurable.

Let $X =$ with Lebesgue measure and $A = [a, b]$. We shall use the familiar notation

$$\int_a^b f(x)\, dx$$

to denote the Riemann integral of f on $[a, b]$ to distinguish it from the Lebesgue integral

$$\int_A f\, d\mu$$

Theorem 17 *a) If f is Riemann integrable on $A = [a, b]$, then $f \in (A)$ and*

$$\int_A f\, d\mu = \int_a^b f(x)\, dx$$

b) Suppose f is bounded on $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if f is continuous almost everywhere on $[a, b]$.

First recall that a step function g on $[a, b]$ is constant on the subintervals of a partition $\{x_0, x_1, \ldots, x_n\}$ of $[a, b]$,

$$g(x) = c_i, \quad \forall x \in (x_i, x_{i-1})$$

and therefore can be represented as a simple measurable function,

$$g = \sum_{i=1}^{m} c_i \chi_{(x_{i-1}, x_i)} + \sum_{i=0}^{m} g(x_i) \chi_{\{x_i\}}$$

(The values of g at the partition points are irrelevant, but we include them here in the second sum for completeness.) Hence the Riemann integral of g is the same as the Lebesgue integral,

$$\begin{aligned} \int_a^b g(x)\, dx &= \sum_{i=1}^{m} c_i (x_i - x_{i-1}) \\ &= \sum_{i=1}^{m} c_i \mu((x_{i-1}, x_i)) + \sum_{i=0}^{m} g(x_i) \mu(\{x_i\}) = \int_A g\, d\mu \end{aligned}$$

If f is Riemann integrable on $[a, b]$, then it is bounded and there are sequences of step functions, $\{s_n\}$, $\{t_n\}$, such that

$$s_n(x) \le f(x) \le t_n(x), \quad \forall\, x \in [a, b]$$

and

$$\lim_{n\to\infty} \int_a^b s_n(x) dx = \int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b t_n(x)\, dx$$

By replacing s_n with $\max\{s_n, s_{n-1}\}$ and t_n with $\min\{t_n, t_{n-1}\}$ for $n > 1$, we may assume s_n is monotonically increasing and t_n is monotonically decreasing. Thus, the limit functions

$$s(x) = \lim_{n\to\infty} s_n(x), \quad t(x) = \lim_{n\to\infty} t_n(x)$$

exist and are measurable by Theorem **??**. By the Lebesgue Dominated Convergence Theorem **??**,

$$\int_A s\, d\mu = \lim_{n\to\infty} \int_A s_n\, d\mu = \lim_{n\to\infty} \int_a^b s_n(x)\, dx = \int_a^b f(x)\, dx$$

and

$$\int_A t\, d\mu = \lim_{n\to\infty} \int_A t_n\, d\mu = \lim_{n\to\infty} \int_a^b t_n(x)\, dx = \int_a^b f(x)\, dx$$

In particular,

$$\int_A s\, d\mu = \int_a^b f(x)\, dx = \int_A t\, d\mu$$

Since $s \leq t$, we conclude from Theorem **??** that $s = t$ almost everywhere on $[a, b]$. But $s \leq f \leq t$ on $[a, b]$, so we also obtain

$$s = f = t \quad \text{almost everywhere on } [a, b] \tag{4}$$

Therefore f is measurable and

$$\int_A f \, d\mu = \int_A s \, d\mu = \int_a^b f(x) \, dx$$

Now suppose x is not one of the partition points for any of the step functions t_n and s_n—the union of these partition points is countable and hence has measure 0. It is then easy to see that $t(x) = s(x)$ if and only if f is continuous at x. Therefore, if f is Riemann integrable on $[a, b]$, (**??**) shows that f is continuous almost everywhere on $[a, b]$.

Conversely, suppose f is bounded on $[a, b]$ and continuous on $[a, b] \setminus B$ where $\mu(B) = 0$. For each $n \in$, let

$$x_i = a + i(b - a)/n, \quad 0 \leq i \leq n$$

and define step functions as follows. For $x \in [x_{i-1}, x_i)$

$$s_n(x) = \inf\{f(t) \mid t \in (x_{i-1}, x_i)\} \leq f(x)$$

$$t_n(x) = \sup\{f(t) \mid t \in (x_{i-1}, x_i)\} \geq f(x)$$

and $s_n(b) = t_n(b) = f(b)$. These sequences are bounded and monotone so their limit functions, $s(x) = \lim_{n\to\infty} s_n(x)$, $t(x) = \lim_{n\to\infty} t_n(x)$, are measurable and

$$\lim_{n\to\infty} \int_A s_n \, d\mu = \int_A s \, d\mu$$

$$\lim_{n\to\infty} \int_A t_n \, d\mu = \int_A t \, d\mu$$

with $s(x) \leq f(x) \leq t(x)$, $\forall \, x \in [a, b]$. Moreover, $s = t$ almost everywhere, since f is continuous on $[a, b] \setminus B$ and $\mu(B) = 0$. By Corollary **??**

$$\int_A s \, d\mu = \int_A t \, d\mu$$

In particular,

$$\lim_{n\to\infty} \int_a^b s_n(x) \, dx = \lim_{n\to\infty} \int_a^b t_n(x) \, dx$$

so that the upper and lower integrals of f agree. Therefore, f is Riemann integrable on $[a, b]$.

Example: Let C be the Cantor set (see p.87) and define $f : [0, 1] \to [0, 1]$ by

$$f(x) = \max C \cap [0, x]$$

This function is clearly increasing and bounded, and so must be Riemann integrable. The previous theorem says that f is then continuous almost everywhere. We can verify this directly by noticing that f is constant on the gaps between the points of C and has a jump discontinuity at each point of C. In particular, f is continuous on $[0, 1]$ except on the Cantor set C which has measure 0. Note that the set of discontinuities in this example is uncountable.