Name: ________________________________ **Instructor:** Jeffrey Diller

Math 366: Honors Real Analysis II
Spring Semester 2004
Exam 1
Wednesday, March 3

This examination contains 5 problems. Counting the front cover and blank pages, the exam consists of 6 sheets of paper.

Scores

Question	Possible	Actual
1	25	
2	15	
3	30	
4	15	
5	15	
Total	100	

GOOD LUCK

1. Do each of the following (5 points each).

(a) State Weierstrass' approximation theorem.

(b) Define what it means for a function $f :^n \to^m$ to be *differentiable* at a point $\mathbf{x} \in^n$.

(c) Define the *norm* of a linear transformation $T :^n \to^m$.

(d) State the inverse function theorem.

(e) State the *chain rule* (this semester's version!).

2. Do one of the following two (use the back of the page if necessary, 15 points).

(a) Consider $f :^2 \rightarrow$ given by $f(0,0) = 0$ and

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

otherwise. Show that both partial derivatives of f exist at every point in ^{2} but that f is not continuous at 0.

(b) Consider the vector space $\mathcal{C}([0,1],)$ of continuous functions $f : [0,1] \rightarrow$ with the norm

$$f = \int_0^1 |f(x)|\, dx.$$

Show that the function $T : \mathcal{C}([0,1],) \rightarrow$ given by $T(f) = f(1)$ is a linear transformation with infinite norm.

3. Do two of the following three (15 points each).

(a) Compute the norm of the linear transformation $T :^2 \to^2$ given by $T(x, y) = (x + 2y, y)$.

(b) Consider the system of equations

$$\begin{aligned} x^2 + y^3 + y + 2z &= 0 \\ 2x^5 + 3x + y^2 - z &= 0. \end{aligned}$$

Clearly $x = 0, y = 0, z = 0$ is a solution for the system. Find (in a reasonably systematic way) 'decent' values for x and y that, together with $z = .1$, approximate another solution to this system. Indicate when you're finished what portion of your computation gives good reason to believe that there actually *is* a point $(x, y, .1)$ near $(0, 0, 0)$ solving the system.

(c) Consider the set

$$M = \{(x, y, z) \in^3 : x + y + 2yz = 1, zy^3 - 2x = -2\}.$$

At what points might M fail to be a submanifold of ^{3}? **Also,** describe the tangent space to M at the point $(1, 0, 0)$.

4. Do one of the following two. Feel free to use any results you like from homework, the book, or lecture (15 points).

(a) Let $f :^2 \to$ be a differentiable function satisfying $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ at every point. Show that $f(x, y) = f(x + y, 0)$ for all $(x, y) \in^2$.

(b) Suppose that $a > 0$ is a given real number and that $f : [-a, a] \to$ is a continuous function such that

$$\int_{-a}^{a} f(x)x^{2n}\, dx = 0$$

for every $n \geq$. Prove that f must be an odd function (i.e. $f(-x) = -f(x)$ for every $x \in$). *Hint: use the fact that*

$$f(x) = g(x) + h(x)$$

where $g(x) := \frac{f(x)+f(-x)}{2}$ *is an even function of* x *and* $h(x) := \frac{f(x)-f(-x)}{2}$ *is an odd function of* x.

5. Do one of the problems you skipped (your choice, 15 points).