Name: ________________________________ **Instructor:** Jeffrey Diller

Math 366: Honors Real Analysis II
Spring Semester 2004
Final Exam
Wednesday, May 5

This examination contains 5 problems. Counting the front cover and blank pages, the exam consists of 6 sheets of paper.

Scores

Question	Possible	Actual
1	35	
2	40	
3	15	
4	10	
5	10	
6	15	
7	15	
8	10	
Total	140	

GOOD LUCK

1. Do all of the following five (7 points each).

(a) Let $E \subset$ be a set. Define the *outer measure* of E.

(b) State the *dominated convergence theorem.*

(c) Define *measurable subset of* .

(d) Define the linear space L^1 (i.e. $L^1()$) together with its norm.

(e) State Fejer's Theorem—define any relevant Fourier series notation.

2. On the next page give examples of four of the following five (10 points each). To define a function, it is enough to draw its graph.

(a) A sequence of continuous functions $f_n :\rightarrow$ that

- f_n converges pointwise but not uniformly to a continuous function $f :\rightarrow$.
- $\int f = \lim \int f_n$.

(b) Same as previous, except that $\int f \neq \lim \int f_n$.

(c) A continuous non-negative integrable function $f :\rightarrow$ such that $\limsup_{x\rightarrow\infty} f(x) = \infty$.

(d) A sequence of functions $f_n :\rightarrow$ such that

- $\lim_{n\rightarrow\infty} f_n = 0$ in L^1;
- $\lim_{n\rightarrow\infty} f_n(x)$ does not exist at any point in .

It suffices here to define the first several f_n's—enough to see the pattern.

(e) A sequence of sets $E_n \subset$ such that $E_{n+1} \subset E_n$ and $mE_n = \infty$ for every $n \in$, but $m(\bigcap_{n\in} E_n) = 0$.

3. Compute the fourier coefficients of the function $f : [-\frac{1}{2}, \frac{1}{2}] \to$ given by $f(x) = |x|$ (15 points).

4. Show that a set $E \subset$ with $m^*E = 0$ is measureable. (10 points)

5. State and prove Fatou's lemma (10 points).

6. Let $f : [-\frac{1}{2}, \frac{1}{2}] \to$ be an L^2 function. Prove that among all trig polynomials of the form $\sum_{n=-N}^{N} c_n e_n(x)$, the fourier approximation $S_N f$ is the best approximation as measured by the L^2 metric (15 points).

7. Let $f :\to$ be a non-negative integrable function and $F :\to$ be given by

$$F(x) = \int_{(-\infty,x)} f.$$

Show that F is continuous (15 points).

8. For ten points extra credit, do one of the following two:

(a) Let $f :\to$ be a 1-periodic, C^2 function. Show that the fourier approximations $S_N f$ converge uniformly to f as $N \to \infty$. (Hint: what is the relationship between fourier coefficients of f and f''.)

(b) Let $f_n : [0, 1] \to$ be measurable functions converging pointwise to a function $f : [0, 1] \to$. Show that for any $\epsilon > 0$, there exists $A \subset [0, 1]$ such that $mA > 1 - \epsilon$ and $f_n \to f$ uniformly on A.