**Beals, page 25, #2:**

**(a)** Note that for $n, x \geq 0$, we have

$$e^{nx} = \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} \leq \frac{n^2x^2}{2}.$$

Therefore, $n^2x^2/e^{nx} \leq 2$ for all $n \in$ and all $x \in [0,1]$. In other words, for $x \in [0,1]$ we have $|n^2x^2/e^{nx}| \leq g(x)$, where $g(x) := 2 \cdot \mathbf{1}_{[0,1]}$.

Moreover, $\lim_{n\to\infty} n^2x^2/e^{nx} \to 0$ for every $x \in [0,1]$ (use L'hôpital's rule, differentiating with respect to $n$, for instance). So we can apply the dominated convergence theorem to conclude

$$\lim_{n\to\infty} \int \frac{n^2x^2}{e^{nx}} = \int \lim_{n\to\infty} \frac{n^2x^2}{e^{nx}} = \int 0 = 0.$$

**(c)** Since $\log(1+t) \leq t$ for $t \geq 0$ and $e^s$ is increasing in $s$, we have

$$\left(1 + \frac{x}{n}\right)^n = e^{n \log\left(1+\frac{x}{n}\right)} \leq e^{n\cdot(x/n)} = e^x.$$

Hence for all $n \in$,

$$\left(1 + \frac{x}{n}\right)^n e^{-\alpha x} \leq e^{(1-\alpha)x},$$

where the right hand side is integrable on $[0, \infty)$ for $\alpha > 1$. It follows from this and the dominated convergence theorem that

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^n e^{-\alpha x} = \int_0^\infty \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n e^{-\alpha x} = \int_0^\infty e^{(1-\alpha)x} = \frac{1}{\alpha - 1}.$$