Beals, page 25 #3: Note that since f is integrable and $F(y) - F(x) \leq \int_x^y |f|$, there is no loss of generality in what follows if we assume that $f \geq 0$.

(a) Let $\{x_n\} \subset$ be any sequence converging to x. Let I_n be the closed interval of points between x_n and x. Then

$$F(x_n) - F(x) = \int_{I_n} f = \int \mathbf{1}_{I_n} f.$$

Now the function $g_n := \mathbf{1}_{I_n} f$ satisfies $|g_n| \leq g := |f|$ for all $n \in$. Moreover, $\lim_{n\to\infty} g_n(y) = 0$ for all $y \neq x$. Therefore,

$$\lim_{n\to\infty} F(x_n) - F(x) = \lim_{n\to\infty} \int g_n = \int \lim_{n\to\infty} g_n = \int 0 = 0.$$

In other words $\lim_{n\to\infty} F(x_n) = F(x)$, which means that F is continuous at x.

(b) F must be uniformly continuous, too. To see this, note that (since $f \geq 0$) F increases from $\lim_{x\to-\infty} F(x) = 0$ to

$$M := \lim_{x\to\infty} F(x) = \int f < \infty.$$

So if $\epsilon > 0$ is given, then there is some number T such that $M - \epsilon/2 < F(x) < M$ for all $x \geq T$ and similarly $0 < F(x) < \epsilon/2$ for all $x < -T$. In particular, if $x, y > T$ or $x, y < -T$, we have $|F(x) - F(y)| < \epsilon/2$.

Moreover, F is uniformly continuous on the compact set $[-T, T]$, so there is $\delta > 0$ such that $x, y \in [0, T]$ and $|x - y| < \delta$ implies that

$$|F(x) - F(y)| < \epsilon/2.$$

Now if by chance $|x - y| < \delta$ and, say, $|x| < T$ while $|y| > T$, we have either $-T$ or T between x and y—say, for argument's sake it's T. Then

$$|F(x) - F(y)| \leq |F(x) - F(T)| + |F(T) - F(y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

So in all cases, $|x - y| < \delta$ implies that $|F(x) - F(y)| < \epsilon$, and F is uniformly continuous.