Rudin, page 165/20:

Let $P(x) = c_n x^n + \ldots + c_1 x + c_0$ be a polynomial. Then

$$\int_0^1 f(x)P(x)\,dx = c_n \int_0^1 f(x)x^n\,dx + \ldots c_1 \int_0^1 f(x)x\,dx + c_0 \int_0^1 f(x)x^0\,dx = 0$$

by hypothesis. By Weierstrass' Approximation Theorem, there is a sequence $\{P_n\}$ of polynomials converging uniformly to f on $[0,1]$. Moreover, f is continuous and therefore bounded on $[0,1]$, as is each of the polynomials P_n. Therefore (see problem 2 from this section), $f \cdot P_n$ converges uniformly to $f \cdot f = f^2$ on $[0,1]$. By Theorem 7.16, I conclude that

$$0 = \lim_{n\to\infty} \int_0^1 f(x)P_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f(x)P_n(x)\,dx = \int_0^1 [f(x)]^2\,dx.$$

Now f^2 is a non-negative function that vanishes exactly where f does. So if $f(x) > 0$ for some $x \in [0,1]$, it follows from continuity of f that for some $\delta > 0$, $|t-x| < \delta$ implies that $f(t) > f(x)/2$. Therefore,

$$\int_0^1 [f(x)]^2\,dx \geq \delta[f(x)]^2/4 > 0.$$

This is impossible, so $f \equiv 0$ on $[0,1]$.