Rudin, page 165/21: The identify function $e^{i\theta} \mapsto e^{i\theta}$ belongs to $\mathcal{A}$, vanishes nowhere and is injective. Hence $\mathcal{A}$ is nowhere vanishing and separates points. Nevertheless, I claim that the function $f(e^{i\theta}) = 1/e^{i\theta} = e^{-i\theta}$ is not in the uniform closure of $\mathcal{A}$.

Note first that

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\,d\theta = 2\pi.$$

Now suppose in order to obtain a contradiction that f is in the uniform closure of $\mathcal{A}$. Then for any $\epsilon > 0$ we could find an element $g \in \mathcal{A}$ such that

$$|g(e^{i\theta}) - f(e^{i\theta})| < \epsilon$$

for every $e^{i\theta}$.

$$\left| \int_0^{2\pi} g(e^{i\theta})e^{i\theta}\,d\theta - \int_0^{2\pi} f(e^{i\theta})e^{i\theta}\,d\theta \right| \leq \int_0^{2\pi} |g(e^{i\theta}) - f(e^{i\theta})||e^{i\theta}|\,d\theta \leq 2\pi\epsilon < 2\pi.$$

provided we choose $\epsilon < 1$. In particular,

$$\int_0^{2\pi} g(e^{i\theta})e^{i\theta} \neq 0.$$

On the other hand $g(\theta) = \sum_{n=0}^{N} c_n e^{in\theta}$, so

$$\int_0^{2\pi} g(e^{i\theta})\,d\theta = \sum_{n=0}^{N} c_n \int_0^{2\pi} e^{in+1\theta}\,d\theta = \sum_{k=1}^{N+1} c_{k-1} \left(\int_0^{2\pi} cos(k\theta)\,d\theta + i \int_0^{2\pi} sin(k\theta)\,d\theta \right) = 0.$$

This contradicts the above and proves that f is not in the uniform closure of $\mathcal{A}$.