Rudin, page 165/23:

I proceed by induction to show that for all $n \geq 0$ and $|x| < 1$ that

$$0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$$

and

$$|x| - P_n(x) \leq |x| \left(1 - \frac{|x|}{2}\right)^n.$$

For the moment, let me suppose that these inequalities are proven. By finding roots of the derivative, it is easily shown that the function $h : [0,1] \to$ given by $h(t) = t(1-t/2)^n$ will achieve its maximum at $t = 0, 1$ or $2/(n+1)$ (i.e. at endpoints or critical points). We have $h(0) = 0$, $h(1) = 1/2^n$, and $h(2/n+1) < 2/(n+1)$ (since $h(x) < x$ when $x \in (0,1]$). In any case, $h(t) < 2/(n+1)$ for all t. Thus

$$||x| - P_n(x)| = |x| - P_n(x) \leq h(x) < 2/(n+1)$$

for all $x \in [-1,1]$, and it follows that P_n converges uniformly to $|x|$.

Now I return to the proof of the inequalities asserted earlier. When $n = 0$, we have $P_n(x) = 0$ and $P_{n+1}(x) = x^2/2$, and all the inequalities are easily verified directly. So now I assume that the inequalities have been verified for $n = k$, and I will prove that they hold when $n = k+1$. First of all, we use $0 \leq P_k(x) \leq |x|$ to estimate

$$P_{k+1}(x) = P_k(x) + \frac{x^2 - P_k^2(x)}{2} \geq P_k(x) + \frac{x^2 - |x|^2}{2} = P_k(x).$$

Secondly,

$$|x| - P_{k+1}(x) = [|x| - P_k(x)] \left[1 - \frac{|x| + P_k(x)}{2}\right] \geq [|x| - P_k(x)] \left[1 - \frac{|x| + |x|}{2}\right] \geq 0$$

for $|x| \leq 1$. So $P_{k+1}(x) \leq |x|$. Finally, in the other direction

$$\begin{aligned}
|x| - P_{k+1}(x) &= [|x| - P_k(x)] \left[1 - \frac{|x| + P_k(x)}{2}\right] \\
&\leq [|x| - P_k(x)] \left[1 - \frac{|x|}{2}\right] \\
&\leq |x| \left(1 - \frac{|x|}{2}\right)^k \left[1 - \frac{|x|}{2}\right] \\
&= |x| \left(1 - \frac{|x|}{2}\right)^{k+1},
\end{aligned}$$

where the second inequality comes from the induction hypothesis. This completes the induction step and the proof.