Rudin, page 239/14:

a) For $(x, y) \neq (0, 0)$, a quick computation shows that

$$D_1 f(x, y) = \frac{x^4 + 3x^2y^2}{(x^2 + y^2)^2}, \quad D_2 f(x, y) = \frac{-2x^3 y}{(x^2 + y^2)^2}.$$

Taking $D_1(f(x, y))$, for example, we note that both x^4 and x^2y^2 are smaller than $|(x, y)|^4$. Hence

$$|D_1 f(x, y)| \leq \frac{4|(x, y)|^2}{|(x, y)|^2} = 4$$

for all $(x, y) \in^2 -(0, 0)$. Likewise,

$$|D_2 f(x, y)| \leq \frac{2|(x, y)|^4}{|(x, y)|^4} = 2.$$

Finally, for $(x, y) = (0, 0)$ one computes

$$D_1 f(0, 0) = \lim_{h \to 0} \frac{(h - 0)}{h} = 1$$

and, in the same fashion, $D_2 f(0, 0) = 0$.

b) Let us write $u = (s, t)$. Then

$$\begin{aligned} D_u f(0, 0) &= \lim_{h \to 0} \frac{f(hu) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{h^3 s^3 / (h^2 s^2 + h^2 t^2)}{h} = \frac{s^3}{s^2 + t^2} = s^3, \end{aligned}$$

since u is a *unit* vector. This shows that $D_u f(0, 0)$ exists and, moreover, since $|s| \leq |u| \leq 1$, we have $|D_u f(0, 0)| \leq 1$.

d) Were f actually differentiable at $(0, 0)$, $D_u f$ would be linear in u. But from the formula computed in part b), we see that

$$D_{(0,1)} f(0, 0) + D_{(1,0)} f(0, 0) = 0 + 1 \neq \frac{1}{2} = D_{(1,1)} f(0, 0).$$