Rudin, page 239/16: From the definition of derivative, we have

$$f'(0) = \lim_{h\to 0} \frac{h + 2h^2 \sin(1/h) - 0}{h} = 1 + \lim_{h\to 0} 2h \sin(1/h) = 1$$

since $|2h\sin(1/h)| \leq 2|h| \to 0$ as $h \to 0$. In particular, $f'(0)$ is invertible (i.e. non-zero).

Moreover, for $t \neq 0$, we have

$$f'(t) = 1 + 4t\sin(1/t) - 2\cos(1/t).$$

Hence $|f'(t)| \leq 1 + 4|t| + 2 < 7$ for $t \in (-1, 1)$. So f' is bounded on $(-1, 1)$.

Now suppose that f is actually injective on some neighborhood $I = (-\epsilon, \epsilon)$ of 0. Then because f is continuous, it follows that f is actually strictly monotone—say for the moment that f is strictly increasing. Then at any point $x \in I$, we have

$$f'(x) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} \geq 0$$

because $h > 0$ implies that $x + h > x$ which implies in turn that $f(x + h) > f(x)$.

But in fact f' is not non-negative on I: at any point $x = 1/2n\pi$ we have $f'(x) = 1 + 4 \cdot (1/2n\pi) \cdot 0 - 2 \cdot 1 = -1$. It follows that f cannot be strictly increasing on I.

So it must be that f is strictly *decreasing* on I. As before we conclude that $f'(x) \leq 0$ for every $x \in I$. This contradicts the fact that $f'(0) = 1$, though. So f is not strictly decreasing, therefore not monotone, and therefore not injective on I. Too bad.