**Rudin, page 239/9:** Fix any point $p \in E$. Let

$$K = \{x \in E : f(x) = f(p)\}.$$

I will show that $K = E$, which implies of course that $f$ is constant. Now $p \in K$, so $K \neq \emptyset$. Moreover, $f$ is continuous because it is differentiable, so $K = f^{-1}(f(p))$ is closed (i.e. the inverse image of a closed set by a continuous function is closed). In particular, $E - K$ is open. If I can show that $K$ is also open, then

$$E = K \cup (K - E)$$

will express $E$ as a disjoint union of two open sets. But $E$ is connected by hypothesis, so it will follow that $K - E =$ and therefore $E = K$.

So to summarize, it's enough to show that $K$ is open. Let $x \in K$ be any point. Because $E$ itself is open, we can choose $r > 0$ such that $N_x(r) \subset E$. Let $y \in N_x(r)$ be any other point and

$$h : [0, 1] \to$$

be given by $h(t) = f(tx + (1 - t)y)$. Then $h(1) = f(x) = f(p)$ and $h(0) = f(y)$. Moreover, as the composition of two differentiable functions $h$ itself is differentiable, with

$$h'(t) = Df(tx + (1 - t)y) \cdot \frac{d}{dt}(tx + (1 - t)y) = \mathbf{0} \cdot (x - y) = 0$$

for all $t \in [0, 1]$. It follows (from one variable calculus) that $h$ is constant. In particular $f(y) = h(1) = h(0) = f(p)$. And $y \in N_r(x)$ was arbitrary, so we conclude that $N_r(x) \subset K$. As $x$ was arbitrary, too, it follows that $K$ is open.