Rudin, page 239/17:

a) The range is all ^{2} except $(0,0)$. Given $(s,t) \in^2$, just let $x = \frac{1}{2}\log(s^2+t^2)$ and choose y so that $\cos y = e^{-x}s$ and $\sin y = e^{-x}t$. This can be done since $(e^{-x}s)^2 + (e^{-x}t)^2) = 1$.

b) The Jacobian of f is

$$\det \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} \neq 0$$

for any $(x,y) \in^2$. So f is locally invertible by the inverse function theorem. However, f is not globally invertible since it's not injective: $f(x, y+2n\pi) = f(x,y)$ for every $n \in$.

c)

$$g(s,t) = (\frac{1}{2}\log(s^2+t^2), \tan^{-1}(t/s)),$$

(choosing $\tan^{-1}$ to have range $(-\pi/2, \pi/2)$). Then

$$g'(s,t) = \begin{pmatrix} \frac{s}{s^2+t^2} & \frac{t}{s^2+t^2} \\ \frac{-t}{s^2+t^2} & \frac{s}{s^2+t^2} \end{pmatrix}.$$

So

$$g(f(x,y)) = \begin{pmatrix} e^{-x}\cos y & e^{-x}\sin y \\ -e^{-x}\sin y & e^{-x}\cos y \end{pmatrix} = f'(x)^{-1}.$$

d) For given y, the set

$$\{f(x,y) : x \in\} = \{e^x(\cos y, \sin y) : x \in\}$$

consists of all positive multiples of the fixed vector $(\cos y, \sin y)$. That is, the image of a horizontal line is a ray beginning at $(0,0)$ (but not including this point).

For given x, the set

$$\{f(x,y) : y \in\} = \{e^x(\cos y, \sin y) : y \in\}$$

consists of all points at distance e^x from $(0,0)$. That is, the image of a vertical line is a circle centered at the origin.