Homework Set 1: Solutions

1. Find the operator norm of the linear transformations \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) with matrices

\[
\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Solution: Let \( L \) be the linear transformation corresponding to the first matrix and \( v = (x, y) \) be a vector. Then

\[
L(v) = (4x, -4y) = \sqrt{(4x)^2 + (-4y)^2} = 4\sqrt{x^2 + y^2} = 4v.
\]

Hence \( L(v)/v = 4 \) regardless of \( v \). It follows that \( L = 4 \).

Now let \( L \) be the linear transformation corresponding to the other matrix. Note that

\[
L(tv)/tv = L(v)/v
\]

for any \( t \in \mathbb{R} \). Hence

\[
sup_{v \in \mathbb{R}^2} L(tv)/tv = sup \{ L(v)/v : v = (x, 1), x \in \mathbb{R} \} = sup \sqrt{(x + 1)^2 + x^2 \over x^2 + 1}
\]

(OK, so I’m missing a multiple of the vector \((1, 0)\), but you can check that one yourself, and anyhow I actually do take care of it implicitly below when I let \( x \to \pm \infty \)). Call the function inside the square root \( f(x) \). Then \( lim_{x \to \pm \infty} f(x) = 2 \). Moreover, after differentiating, we see that \( f \) has critical points when

\[
x^2 - x = 0 \Rightarrow x = 1, 0.
\]

Since \( f(1) = 5/2 \) and \( f(0) = 1 \), we conclude that \( L = \sqrt{5}/2 \).

2. Let \( V \) be a vector space over the field \( \mathbb{F} \) (or \( \mathbb{C} \)). A norm on \( V \) is a function \( \cdot \) : \( V \to \mathbb{R} \) such that for all \( \lambda \in \mathbb{R} \) and \( v, w \in V \),

- \( v \geq 0 \) with equality if and only if \( v = 0 \).
- \( \lambda v = |\lambda|v \)
- \( v + w \leq v + w \).

Given a norm \( \cdot \) on \( V \), show that

\[
d(v, w) = v - w
\]

defines a metric on \( V \). A set \( U \) is said to be open with respect to \( \cdot \) if it is open with respect to the associated metric \( d \).

Solution: We first check that \( d \) is a metric. Clearly \( d(v, w) = v - w \geq 0 \), and

\[
v - w = 0 \iff v - w = 0 \iff v = w.
\]

Symmetry of \( d \) follows from \( v - w = |v - w| \). Finally,

\[
d(v, w) = v - w = (v - u) - (w - u) \leq v - u + w - u = d(v, u) + d(u, w),
\]
so the triangle inequality holds. Thus $d$ is a metric.

3. Different norms $\cdot$ and $\cdot'$ on the same vector space are called comparable if there are constants $C_1, C_2 > 0$ such that

$$C_1 \mathbf{v} \leq \mathbf{v}' \leq C_2 \mathbf{v}$$

for all $\mathbf{v} \in V$.

Supposing that $\cdot$, $\cdot'$ are comparable, show that a set $U \subset V$ is open with respect to $\cdot$ if and only if it is open with respect to $\cdot'$. Does the same conclusion hold if you replace ‘open’ with ‘closed’? ‘compact”? ‘connected”? Explain.

**Solution:** Let $U \subset V$ be open with respect to $\cdot$ and $\mathbf{v} \in U$. Then there exists $r > 0$ such that $N_r(\mathbf{v}) = \{\mathbf{w} \in V : \mathbf{w} - \mathbf{v} < r\} \subset U$. But since $\mathbf{w} - \mathbf{v}' \leq r/C_2 \Rightarrow \mathbf{w} - \mathbf{v} \leq r$,

we have $N'_{r/C_2}(\mathbf{v}) \subset N_r(\mathbf{v}) \subset U$ (where the prime denotes ‘neighborhood with respect to $\cdot'$). That is, any $\mathbf{v} \in U$ admits a $\cdot'$ neighborhood also contained in $U$, so $U$ is open with respect to $\cdot'$.

The same argument shows that if $U$ is open with respect to $\cdot'$, then $U$ is also open with respect to $\cdot$. \[\square\]

The conclusion also works for closed sets, compact sets, and connected sets, because all of these can be characterized in terms of open sets (e.g. a set is closed iff it’s the complement of an open set, etc. etc.)

4. Let $n, m \in \mathbb{N}$ be given and $V = L^{(n, m)}$ be the vector space of linear transformations from $n$ to $m$. Let $T = (a_{ij}) \in V$ be an arbitrary element. Show that the following norms on $V$ are all comparable to the operator norm on $V$.

- $[\infty]T = \max_{i,j} |a_{ij}|$
- $[1]T = \sum_{i,j} |a_{ij}|$
- $[2]T = \sqrt{\sum_{i,j} |a_{ij}|^2}$

In fact, it can be shown that pretty much any two norms on a finite dimensional vector space are comparable (Prove this and you take care of all the above items at once. And I’ll give you five extra credit points).

**Solution:** Let $a = \max |a_{ij}|$. Then

$$a = \sqrt{a^2} \leq \sqrt{\sum_{i,j} a_{ij}^2} \leq \sqrt{\left(\sum_{i,j} |a_{ij}|\right)^2} = \sum_{i,j} |a_{ij}| \leq nm a,$$

where $nm$ is just the number of entries in $T$. Since all these inequalities hold regardless of $T$, this shows that $[\infty]$, $[2]$ and $[1]$ are all comparable. To finish the proof it’s enough to show that $\cdot$ is comparable to any one of these—say $[\infty]$.

If $\mathbf{v} = \mathbf{e}_j$ is one of the usual basis vectors, then

$$T(\mathbf{v}) = (a_{1j}, a_{2j}, \ldots, a_{mj}) = \sqrt{\sum_i a_{ij}^2} \leq \sqrt{\sum_i a^2} = \sqrt{ma},$$
and if \( \mathbf{v} = v_1 \mathbf{e}_1 + \ldots + v_n \mathbf{e}_n \) is an arbitrary unit vector, then
\[
T(\mathbf{v}) = \sum_j v_j T(\mathbf{e}_j) \leq \sum_j |v_j| T(\mathbf{e}_j) \leq n \cdot \sqrt{m} a
\]
because \(|v_j| \leq 1\) for all \(j\). Hence
\[
T(\mathbf{v}) = \sup_{\mathbf{v}=1} T(\mathbf{v}) \leq n \sqrt{m} [\infty] T.
\]

\[\square\]

By the way,

**Theorem.** Any two norms on a finite dimensional real (or complex) vector space \(V\) are comparable.

**Proof.** Let \(\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}\) be a basis for \(V\) and \([\infty]\) be the norm on \(V\) given by
\[
[\infty] \mathbf{v} = \max_{1 \leq j \leq n} |c_j|
\]
where the numbers \(c_j\) come from writing \(\mathbf{v} = c_1 \mathbf{e}_1 + \ldots + c_n \mathbf{e}_n\) as a linear combination of basis vectors.
It is enough to show that any other norm \(\cdot\) on \(V\) is comparable to \([\infty]\). Now on the one hand, we have
\[
\mathbf{v} \leq |c_1| \mathbf{e}_1 + \cdots + n(\max \mathbf{e}_j) [\infty] \mathbf{v},
\]
which gives comparability in one direction.

To get comparability in the other direction, I suppose for the sake of obtaining a contradiction that for any \(C > 0\) there exists \(\mathbf{v} \in V\) such that \([\infty] \mathbf{v} > C \mathbf{v}\). Then in particular, by choosing a sequence of \(C\)'s tending to \(\infty\), we can find a sequence of vectors \(\{\mathbf{v}_j\} \subset V\) such that \([\infty] \mathbf{v}_j = 1\) whereas \(\lim_{j \to \infty} \mathbf{v}_j = 0\).

Given this, I claim that after passing to a subsequence, we can further assume that \(\{\mathbf{v}_j\}\) converges to some vector \(\mathbf{v} \in V\). And I never claim anything that I can’t prove. Never. If we write
\[
\mathbf{v}_j = c_{1j} \mathbf{e}_1 + \cdots + c_{nj} \mathbf{e}_n,
\]
then the ‘coordinate vectors’ \((c_{1j}, \ldots, c_{nj}) \in \mathbb{R}^n\) all lie in the compact (because closed and bounded) set \(\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \max |x_k| = 1\}\), so after passing to a subsequence, we can assume that \(c_{1j} \to c_1, \ldots, c_{nj} \to c_n\) where \(\max |c_k| = 1\). But, from the definition of \([\infty]\), this is the same as saying that
\[
\lim_{n \to \infty} [\infty] \mathbf{v}_j - \mathbf{v} = 0
\]
where \(\mathbf{v} = c_1 \mathbf{e}_1 + \cdots + c_n \mathbf{e}_n\). So the claim is true.

We get our contradiction as follows. By the triangle inequality
\[
|\mathbf{v}_j - \mathbf{v}, \mathbf{v} - \mathbf{v}_j| \leq |\mathbf{v}_j - \mathbf{v}|
\]
That is,
\[
|\mathbf{v}_j - \mathbf{v}| \leq |\mathbf{v}_j - \mathbf{v} - C [\infty] \mathbf{v}_j - \mathbf{v} - 0
\]
as \(j \to \infty\). So \(\mathbf{v} = 0\). On the other hand \(\mathbf{v}\) is certainly non-zero, because the basis vectors \(\mathbf{e}_j\) are linearly independent and at least one of the coefficients \(c_j\) used to define \(\mathbf{v}\) has magnitude 1. Since non-zero vectors must have non-zero norm, we have found our impasse and conclude that there really does exist \(C > 0\) such that
\[
[\infty] \mathbf{v} \leq C \mathbf{v}
\]
for every $v \in V$.

5. Give an example of two incomparable norms on the (infinite dimensional) vector space $C([0,1],\mathbb{R})$ of continuous functions from $[0,1]$ to $\mathbb{R}$.

**Solution:** The norms $[\infty]f = \max_{x\in[0,1]} |f(x)|$ and $[1]f = \int_0^1 |f(x)| \, dx$ are incomparable. Consider for instance the functions $f_n(x) = x^n$. We have $[\infty]f_n = |f_n(1)| = 1$ for every $n \in \mathbb{N}$, but $[1]f_n = \frac{1}{n+1} \to 0$. Hence, there is no constant $C > 0$ such that $[\infty]f \leq C[1]f$ for all $f \in C([0,1],\mathbb{R})$. 