Solutions to Homework 3

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Rudin, page 239/9: Fix any point $p \in E$. Let

$$K = \{x \in E : f(x) = f(p)\}.$$

I will show that $K = E$, which implies of course that f is constant. Now $p \in K$, so $K \neq \emptyset$. Moreover, f is continuous because it is differentiable, so $K = f^{-1}(f(p))$ is closed (i.e. the inverse image of a closed set by a continuous function is closed). In particular, $E - K$ is open. If I can show that K is also open, then

$$E = K \cup (K - E)$$

will express E as a disjoint union of two open sets. But E is connected by hypothesis, so it will follow that $K - E =$ and therefore $E = K$.

So to summarize, it's enough to show that K is open. Let $x \in K$ be any point. Because E itself is open, we can choose $r > 0$ such that $N_x(r) \subset E$. Let $y \in N_x(r)$ be any other point and

$$h : [0, 1] \to$$

be given by $h(t) = f(tx + (1 - t)y)$. Then $h(1) = f(x) = f(p)$ and $h(0) = f(y)$. Moreover, as the composition of two differentiable functions h itself is differentiable, with

$$h'(t) = Df(tx + (1 - t)y) \cdot \frac{d}{dt}(tx + (1 - t)y) = \mathbf{0} \cdot (x - y) = 0$$

for all $t \in [0, 1]$. It follows (from one variable calculus) that h is constant. In particular $f(y) = h(1) = h(0) = f(p)$. And $y \in N_r(x)$ was arbitrary, so we conclude that $N_r(x) \subset K$. As x was arbitrary, too, it follows that K is open.