Solutions to Homework 4

[11pt]article graphicx amssymb epstopdf

Rudin, page 239/15:

a) We have $(x^4 + y^2)^2 - 4x^4y^2 = (x^4 - y^2)^2 \geq 0$ for all $(x, y) \in^2$.

To see that f is continuous, it's enough to check continuity at $(0, 0)$:

$$\lim_{(x,y)\to(0,0)} x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4 + y^2)^2} = 0 - \lim_{(x,y)\to(0,0)} x^2 \frac{4x^4y^2}{(x^4 + y^2)^2} = 0$$

since $x^2 \to 0$ whereas the magnitude of the other factor is bounded above by 1. Since $f(0, 0) = 0$ by definition, this shows that f is continuous at $(0, 0)$.

b) A computation shows that

$$g_\theta(t) = t^2 - 2t^3(\cos^2\theta \sin\theta) - t^4 h(t)$$

where

$$h(t) = \frac{4\cos^6\theta \sin^2\theta}{(t^2\cos^4\theta - \sin^2\theta)}$$

is defined and C^∞ even at $t = 0$ (when $\theta \neq 2n\pi$, we have $\sin\theta \neq 0$ and this is clear; when $\theta = 2n\pi$ the numerator vanishes altogether and $h(t) \equiv 0$). Thus

$$\begin{aligned} g'_\theta(t) &= 2t - 6t^2(\cos^2\theta \sin\theta) - t^3(4h(t) + th'(t)) \\ g''_\theta(t) &= 2 - 12t(\cos^2\theta \sin\theta) - t^2(12h(t) + 8th'(t) + t^2h''(t)) \end{aligned}$$

It follows that $g'_\theta(0) = 0$, $g''_\theta(0) = 2$. Thus $g_\theta(t)$ has a strict local minimum at $t = 0$.

However, it doesn't follow that f has a local minimum at $(0, 0)$. In fact,

$$f(x, x^2) = x^2 + x^4 - 2x^4 - \frac{4x^{10}}{(2x^4)^2} = x^2 - x^4 + x^2 = -x^4,$$

so that e.g. $\{(1/n, 1/n^2)\}$ is a sequence of points converging to $(0, 0)$ such that $f(1/n, 1/n^2) < 0 = f(0, 0)$ for every $n \in$.