Solutions to Homework 7

Beals page 5, #4: The inequality

$$m^*(A_1 \cup A_2) \leq m^* A_1 + m^* A_2$$

holds for *all* $A_1, A_2 \subset$, so it remains for me to use the hypothesis to prove the reverse inequality.

Given $\epsilon > 0$, let $\mathcal{I}$ be a finite or countable collections of intervals covering $A_1 \cup A_2$ and satisfying

$$|\mathcal{I}| \leq m^*(A_1 \cup A_2) + \epsilon.$$

For each interval $I \in \mathcal{I}$, let $I' = I \cap I_1$ and $I'' = I \cap I_2$. Note that

- For any $I \in \mathcal{I}$, the corresponding sets I' and I'' are open intervals satisfying

  $$|I'| + |I''| \leq |I|.$$

  .

- $\mathcal{I}'\{I' : I \in \mathcal{I}\}$ covers A_1
- $\mathcal{I}''\{I'' : I \in \mathcal{I}\}$ covers A_2.

Therefore,

$$m^* A_1 + m^* A_2 \leq |\mathcal{I}'| + |\mathcal{I}''| \leq |\mathcal{I}| \leq m^*(A_1 \cup A_2) + \epsilon.$$

But $\epsilon > 0$ was arbitrary, so I deduce that

$$m^* A_1 + m^* A_2 \leq m^*(A_1 \cup A_2)$$

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Beals page 5, #7: The trick here is to remove smaller and smaller fractions of the remaining intervals as the construction progresses. Specifically, I define a decreasing sequence $\{C_k\}$ of closed sets $C_k \subset [0,1]$ as follows. First I choose a sequence $\{a_j\}_{j\in}$ of positive real numbers such that

$$s := \sum_{j=1}^{\infty} a_j < 1.$$

I let $C_0 = [0,1]$ and $D_0 = [0,1] - C_0 = \emptyset$. I divide C_0 into two closed intervals of equal length by removing an open interval of length a_1 centered at $1/2$. I call the union of the remaining closed intervals C_1 and set $D_1 = [0,1] - C_1$.

Likewise, given a closed set $C_k \subset [0,1]$ consisting of 2^k closed, pairwise disjoint intervals I of equal length, I create the set $C_{k+1} \subset C_k$ by removing from each interval $I \subset C_k$ an open interval J centered on the midpoint of I such that $|J| \leq a_{k+1}|I|$. Thus, C_{k+1} consists of 2^{k+1} closed, pairwise disjoint intervals. Moreover, since the sum of the lengths of the closed intervals comprising C_k is no greater than one, it follows that the sum of the lengths of the intervals removed from C_k to create C_{k+1} is no larger than a_{k+1}. Stated in terms of the complements D_k and D_{k+1} of C_k and C_{k+1} in $[0, 1]$, I have

$$m^* D_{k+1} \leq m^* D_k + a_{k+1}.$$

Now if I let $C = \bigcap_{k\in} C_k$ and $D = \bigcup_{k\in} D_k$, I have $C \cup D = [0, 1]$. Hence

$$m^* C \geq 1 - m^* D \geq 1 - \sum_{k=0}^{\infty} m^*(D_k - D_{k-1}) \geq 1 - \sum_{k=0}^{\infty} a_k = 1 - s > 0.$$