Math 366, Assignment 1
Profs personal problems (assigned 1/16/03)

(1) Find the operator norm of the linear transformations $L : \mathbf{R}^2 \to \mathbf{R}^2$ with matrices

$$\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(2) Let V be a vector space over the field $\mathbf{R}$ (or $\mathbf{C}$). A *norm* on V is a function $\|\cdot\| : V \to \mathbf{R}$ such that for all $\lambda \in \mathbf{R}$ and $\mathbf{v}, \mathbf{w} \in V$,

- $\|\mathbf{v}\| \geq 0$ with equality if and only if $\mathbf{v} = 0$.
- $\|\lambda \mathbf{v}\| = |\lambda| \, \|\mathbf{v}\|$
- $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

Given a norm $\|\cdot\|$ on V, show that

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$

defines a metric on V. A set U is said to be *open with respect to* $\|\cdot\|$ if it is open with respect to the associated metric d.

(3) Different norms $\|\cdot\|$ and $\|\cdot\|'$ on the same vector space are called *comparable* if there are constants $C_1, C_2 > 0$ such that

$$C_1 \|\mathbf{v}\| \leq \|\mathbf{v}\|' \leq C_2 \|\mathbf{v}\|$$

for all $\mathbf{v} \in V$.

Supposing that $\|\cdot\|, \|\cdot\|'$ are comparable, show that a set $U \subset V$ is open with respect to $\|\cdot\|$ if and only if it is open with respect to $\|\cdot\|'$. Does the same conclusion hold if you replace 'open' with 'closed'? 'compact'? 'connected'? Explain.

(4) Let $n, m \in \mathbf{Z}^+$ be given and $V = L(\mathbf{R}^n, \mathbf{R}^m)$ be the vector space of linear transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$. Let $T = (a_{ij}) \in V$ be an arbitrary element. Show that the following norms on V are all comparable to the operator norm on V.

- $\|T\|_\infty = \max_{i,j} |a_{ij}|$
- $\|T\|_1 = \sum_{i,j} |a_{ij}|$
- $\|T\|_2 = \sqrt{\sum_{i,j} |a_{ij}|^2}$

In fact, it can be shown that pretty much any two norms on a finite dimensional vector space are comparable (Prove this and you take care of all the above items at once. And I'll give you five extra credit points).

(5) Show that the norms

- $\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$
- $\|f\|_1 = \int_0^1 |f(x)| \, dx$

on the (infinite dimensional) vector space $C([0,1], \mathbf{R})$ are not comparable.