Solutions to Homework 7

Beals page 5, #4: The inequality

$$m^*(A_1 \cup A_2) \leq m^* A_1 + m^* A_2$$

holds for *all* $A_1, A_2 \subset \mathbf{R}$, so it remains for me to use the hypothesis to prove the reverse inequality.

Given $\epsilon > 0$, let $\mathcal{I}$ be a finite or countable collections of intervals covering $A_1 \cup A_2$ and satisfying

$$|\mathcal{I}| \leq m^*(A_1 \cup A_2) + \epsilon.$$

For each interval $I \in \mathcal{I}$, let $I' = I \cap I_1$ and $I'' = I \cap I_2$. Note that

- For any $I \in \mathcal{I}$, the corresponding sets I' and I'' are open intervals satisfying

$$|I'| + |I''| \leq |I|.$$

.

- $\mathcal{I}' := \{I' : I \in \mathcal{I}\}$ covers A_1
- $\mathcal{I}'' := \{I'' : I \in \mathcal{I}\}$ covers A_2.

Therefore,

$$m^* A_1 + m^* A_2 \leq |\mathcal{I}'| + |\mathcal{I}''| \leq |\mathcal{I}| \leq m^*(A_1 \cup A_2) + \epsilon.$$

But $\epsilon > 0$ was arbitrary, so I deduce that

$$m^* A_1 + m^* A_2 \leq m^*(A_1 \cup A_2)$$

Beals page 5, #7: The trick here is to remove smaller and smaller fractions of the remaining intervals as the construction progresses. Specifically, I define a decreasing sequence $\{C_k\}$ of closed sets $C_k \subset [0,1]$ as follows. First I choose a sequence $\{a_j\}_{j \in \mathbf{N}}$ of positive real numbers such that

$$s := \sum_{j=1}^{\infty} a_j < 1.$$

I let $C_0 = [0,1]$ and $D_0 = [0,1] - C_0 = \emptyset$. I divide C_0 into two closed intervals of equal length by removing an open interval of length a_1 centered at $1/2$. I call the union of the remaining closed intervals C_1 and set $D_1 = [0,1] - C_1$.

Likewise, given a closed set $C_k \subset [0,1]$ consisting of 2^k closed, pairwise disjoint intervals I of equal length, I create the set $C_{k+1} \subset C_k$ by removing from each interval $I \subset C_k$ an open interval J centered on the midpoint of I such that $|J| \leq a_{k+1}|I|$. Thus, C_{k+1} consists of 2^{k+1} closed, pairwise disjoint intervals. Moreover, since the sum of the lengths of the closed intervals comprising C_k is no greater than one, it follows that the sum of the lengths of the intervals removed from C_k to create C_{k+1} is no larger than a_{k+1}. Stated in terms of the complements D_k and D_{k+1} of C_k and C_{k+1} in $[0,1]$, I have

$$m^*D_{k+1} \le m^*D_k + a_{k+1}.$$

Now if I let $C = \bigcap_{k\in\mathbf{N}} C_k$ and $D = \bigcup_{k\in\mathbf{N}} D_k$, I have $C \cup D = [0,1]$. Hence

$$m^*C \ge 1 - m^*D \ge 1 - \sum_{k=0}^{\infty} m^*(D_k - D_{k-1}) \ge 1 - \sum_{k=0}^{\infty} a_k = 1 - s > 0.$$

Beals page 5, #9: Note that m^*A is finite because A is bounded. Suppose, in order to obtain a contradiction, that $m^*A \neq 0$. Then (by setting $\epsilon = m^*A > 0$), I can find a countable collection $\mathcal{I} = \{I_k\}_{k\in\mathbf{N}}$ of open intervals covering A such that $|\mathcal{I}| < m^*A + \epsilon = 2m^*A$. But then

$$m^*A \ge \sum_{k=1}^{\infty} m^*(A \cap I_k) \le \frac{1}{2}\sum_{k=1}^{\infty} |I_k| = \frac{1}{2}|\mathcal{I}| < m^*A.$$

This contradiction proves that $m^*A = 0$. □

Beals page 11, #1: If $E \subset \mathbf{R}$ is any set, then

$$m^*E \le m^*(E \cap A) + m^*(E \cap A^c)$$

automatically, regardless of m^*A. On the other hand, since $m^*A = 0$,

$$m^*(E \cap A) + m^*(E \cap A^c) \le m^*A + m^*E = m^*E.$$

Hence $m^*E = m^*(E \cap A) + m^*(E \cap A^c)$ for all $E \subset \mathbf{R}$. That is, A is measurable. □

Beals page 11, #4: For example, $A_n = [n, \infty)$.

Beals page 11, #6:

a) Let $A_n = [0,1]$ if $n \in \mathbf{N}$ is even and $A_n = \emptyset$ if n is odd. Then $\limsup A_n = [0,1]$ and $\liminf A_n = \emptyset$.

b) I show only that $\limsup A_n$ is measurable. The proof for $\liminf A_n$ is similar. For each $N \in \mathbf{N}$, let

$$B_N = \bigcup_{n\ge N} A_n$$

Then B_N and $B := \cap_{N\in\mathbf{N}} B_N$ are measurable by assertion I on page 9 of Beals notes.

I claim that $B = \limsup A_n$. To see that this is so, let $x \in B$. Then $x \in B_N$ for every $N \in \mathbf{N}$. In other words, for each $N \in \mathbf{N}$, there exists $n \ge N$ such that $x \in A_n$. This can only be the case if $x \in A_n$ for infinitely many n (otherwise, we could let N be one larger than the maximum of those finitely many n for which $x \in A_n$, and it would follow that $x \notin B_N$). So $x \in \limsup A_n$.

In the other direction, suppose that $x \notin B$. Then $x \notin B_N$ for some $N \in \mathbf{N}$. Then $x \in A_n$ for at most N values of n. It follows that $x \notin \limsup A_n$. This and the preceding paragraph show that $x \in B$ if and only if $x \in \limsup A_n$, so the two sets are equal, and I conclude that $\limsup A_n$ is measurable. □

Beals page 11, #7: The function $d(A, B)$ is symmetric in A and B, because $A \triangle B = B \triangle A$.

For transitivity, consider sets $A, B, C \subset \mathbf{R}$ and let $x \in A \triangle C$ be any element. Say for instance (and with no loss of generality) that $x \in A$ but $x \notin C$. Then if $x \in B$, it follows that $x \in B \triangle C$; and if $x \notin B$, it follows that $x \in A \triangle B$. Either way, $x \in A \triangle B \cup B \triangle C$. This proves that

$$A \triangle C \subset A \triangle B \cup B \triangle C,$$

Consequently,

$$m(A \triangle C) \leq m(A \triangle B) + m(B \triangle C).$$

It follows that d is transitive and a semi-metric. □