Solutions to Homework 8

Beals, page 22 #9: No, it's not always true. Consider for example the function $f : \mathbf{R} \to \mathbf{R}$ which is zero everywhere except on intervals of the form $[n - 1/n^2, n + 1/n^2]$, $n \in \mathbf{N}$; and on such an interval the graph of f is the pair of lines joining the points $(n - 1/n^2, 0)$ and $(n + 1/n^2, 0)$ to the point $(n, 1)$. Then f is continuous and $f(n) = 1$ for every $n \in \mathbf{N}$, so $\lim_{x\to\infty} f(x) \neq 0$ (in fact, the limit doesn't exist). However,

$$\int f = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Beals, page 25, #2:

(a) Note that for $n, x \geq 0$, we have

$$e^{nx} = \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} \leq \frac{n^2x^2}{2}.$$

Therefore, $n^2x^2/e^{nx} \leq 2$ for all $n \in \mathbf{N}$ and all $x \in [0, 1]$. In other words, for $x \in [0, 1]$ we have $|n^2x^2/e^{nx}| \leq g(x)$, where $g(x) := 2 \cdot \mathbf{1}_{[0,1]}$.

Moreover, $\lim_{n\to\infty} n^2x^2/e^{nx} \to 0$ for every $x \in [0, 1]$ (use L'hôpital's rule, differentiating with respect to n, for instance). So we can apply the dominated convergence theorem to conclude

$$\lim_{n\to\infty} \int \frac{n^2x^2}{e^{nx}} = \int \lim_{n\to\infty} \frac{n^2x^2}{e^{nx}} = \int 0 = 0.$$

(c) Since $\log(1 + t) \leq t$ for $t \geq 0$ and e^s is increasing in s, we have

$$\left(1 + \frac{x}{n}\right)^n = e^{n \log\left(1+\frac{x}{n}\right)} \leq e^{n\cdot(x/n)} = e^x.$$

Hence for all $n \in \mathbf{N}$,

$$\left(1 + \frac{x}{n}\right)^n e^{-\alpha x} \leq e^{(1-\alpha)x},$$

where the right hand side is integrable on $[0, \infty)$ for $\alpha > 1$. It follows from this and the dominated convergence theorem that

$$\lim_{n\to\infty} \int_0^{\infty} \left(1 + \frac{x}{n}\right)^n e^{-\alpha x} = \int_0^{\infty} \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n e^{-\alpha x} = \int_0^{\infty} e^{(1-\alpha)x} = \frac{1}{\alpha - 1}.$$

Beals, page 25 #3: Note that since f is integrable and $F(y) - F(x) \leq \int_x^y |f|$, there is no loss of generality in what follows if we assume that $f \geq 0$.

(a) Let $\{x_n\} \subset \mathbf{R}$ be any sequence converging to x. Let I_n be the closed interval of points between x_n and x. Then

$$F(x_n) - F(x) = \int_{I_n} f = \int \mathbf{1}_{I_n} f.$$

Now the function $g_n := \mathbf{1}_{I_n} f$ satisfies $|g_n| \leq g := |f|$ for all $n \in \mathbf{N}$. Moreover, $\lim_{n\to\infty} g_n(y) = 0$ for all $y \neq x$. Therefore,

$$\lim_{n\to\infty} F(x_n) - F(x) = \lim_{n\to\infty} \int g_n = \int \lim_{n\to\infty} g_n = \int 0 = 0.$$

In other words $\lim_{n\to\infty} F(x_n) = F(x)$, which means that F is continuous at x. □

(b) F must be uniformly continuous, too. To see this, note that (since $f \geq 0$) F increases from $\lim_{x\to-\infty} F(x) = 0$ to

$$M := \lim_{x\to\infty} F(x) = \int f < \infty.$$

So if $\epsilon > 0$ is given, then there is some number T such that $M - \epsilon/2 < F(x) < M$ for all $x \geq T$ and similarly $0 < F(x) < \epsilon/2$ for all $x < -T$. In particular, if $x, y > T$ or $x, y < -T$, we have $|F(x) - F(y)| < \epsilon/2$.

Moreover, F is uniformly continuous on the compact set $[-T, T]$, so there is $\delta > 0$ such that $x, y \in [0, T]$ and $|x - y| < \delta$ implies that

$$|F(x) - F(y)| < \epsilon/2.$$

Now if by chance $|x - y| < \delta$ and, say, $|x| < T$ while $|y| > T$, we have either $-T$ or T between x and y—say, for argument's sake it's T. Then

$$|F(x) - F(y)| \leq |F(x) - F(T)| + |F(T) - F(y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

So in all cases, $|x-y| < \delta$ implies that $|F(x)-F(y)| < \epsilon$, and F is uniformly continuous.
□

Beals, page 25 #5: Set $g_n(x) = \inf_{j\geq n} f_j(x)$. Then for all $n \in \mathbf{N}$ and $x \in \mathbf{R}$,

- $\liminf f_n(x) = \lim g_n(x)$,
- $0 \leq g_n(x) \leq f_n(x)$,
- $g_{n+1}(x) \geq g_n(x)$

The second and third items allow us to apply the monotone convergence theorem to g_n and obtain

$$\int \liminf f_n = \int \lim g_n = \lim \int g_n = \liminf \int g_n \leq \liminf \int f_n.$$

Beals, page 25 #6: Let $f_n : \mathbf{R} \to \mathbf{R}$ be the 'tent' function.

$$f_n(x) = \begin{cases} n^2x & \text{if} \quad 0 \le x \le 1/n \\ 2n - n^2x & \text{if} \quad 1/n \le x \le 2/n \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\liminf \int f_n(x) = \liminf 1 > 0 = \int 0 = \int \lim f_n = \int \liminf f_n.$$

Beals, page 27 #1: By the monotone convergence theorem, we have

$$\infty > \lim \int f_n = \int \lim f_n = \int f.$$

So by the proposition on page 27, it follows that f is finite a.e. □

Beals, page 34 #2: The assertion is false. To see this, consider the function which is zero except on closed intervals $[n - 1/4^n, n + 1/4^n]$, $n \ge 2$. And to define f on each of these intervals take $f(n) = 2^n$ and then make f linear on each of the remaining subintervals (i.e. the graph of f on the interval is a triangle of height 2^n and width $2/4^n$). Then on the one hand $\lim_{n\to\infty} f(n) = \lim 2^n = \infty$, so f is not bounded. But on the other hand,

$$\int f = \sum_{n=2}^{\infty} \frac{1}{2}\frac{2}{4^n}2^n = \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2} < \infty.$$

Beals, page 34 #3: Let $\epsilon > 0$ be given. By Theorem 2 on page 33, there is a continuous, compactly supported function g such that $\|g - f\|_1 < \epsilon/4$. Note that in fact $\|g_a - f_a\|_1 = \|g - f\|_1 < \epsilon/4$ for all $a \in \mathbf{R}$.

Let $M > 0$ be chosen so that $g \equiv 0$ outside $[-M, M]$. Because g has compact support, it is uniformly continuous. So I can choose $\delta > 0$ so that $|x - y| < \delta$ implies that $|g(x) - g(y)| < \epsilon/8M$. I can, of course, assume that $\delta < M$, too. Therefore, if $|a| < \delta$, it follows that $g_a \equiv 0$ outside $[-2M, 2M]$ and

$$\begin{aligned} \|f - f_a\|_1 &\le \|f - g\|_1 + \|g - g_a\|_1 + \|f_a - g_a\|_1 \\ &< \frac{\epsilon}{4} + \int_{-2M}^{2M} (g(x) - g(x - a)) + \frac{\epsilon}{4} \\ &< \frac{\epsilon}{2} + \int_{-2M}^{2M} \frac{\epsilon}{8M} = \epsilon. \end{aligned}$$

This proves that $\lim_{a\to 0} \|f - f_a\|_1 = 0$. □