Homework Set 1: Solutions

1. Find the operator norm of the linear transformations $L : \mathbf{R}^2 \to \mathbf{R}^2$ with matrices

$$\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solution: Let L be the linear transformation corresponding to the first matrix and $\mathbf{v} = (x, y)$ be a vector. Then

$$\|L(\mathbf{v})\| = \|(4x, -4y)\| = \sqrt{(4x)^2 + (-4y)^2} = 4\sqrt{x^2 + y^2} = 4\,\|\mathbf{v}\|.$$

Hence $\|L(\mathbf{v})\| / \|\mathbf{v}\| = 4$ regardless of $\mathbf{v}$. It follows that $\|L\| = 4$.

Now let L be the linear transformation corresponding to the other matrix. Note that

$$\|L(t\mathbf{v})\| / \|t\mathbf{v}\| = \|L(\mathbf{v})\| / \|\mathbf{v}\|$$

for any $t \in \mathbf{R}$. Hence

$$\begin{aligned} \sup_{\mathbf{v}\in\mathbf{R}^2} \|L(t\mathbf{v})\| / \|t\mathbf{v}\| &= \sup\{\|L(\mathbf{v})\| / \|\mathbf{v}\| : \mathbf{v} = (x, 1), x \in \mathbf{R}\} \\ &= \sup_{x\in\mathbf{R}} \sqrt{\frac{(x+1)^2 + x^2}{x^2 + 1}} \end{aligned}$$

(OK, so I'm missing a multiple of the vector $(1, 0)$, but you can check that one yourself, and anyhow I actually do take care of it implicitly below when I let $x \to \pm\infty$.). Call the function inside the square root $f(x)$. Then $\lim_{x\to\pm\infty} f(x) = 2$. Moreover, after differentiating, we see that f has critical points when

$$x^2 - x = 0 \Rightarrow x = 1, 0.$$

Since $f(1) = 5/2$ and $f(0) = 1$, we conclude that $\|L\| = \sqrt{5/2}$.

2. Let V be a vector space over the field $\mathbf{R}$ (or $\mathbf{C}$). A *norm* on V is a function $\|\cdot\| : V \to \mathbf{R}$ such that for all $\lambda \in \mathbf{R}$ and $\mathbf{v}, \mathbf{w} \in V$,

- $\|\mathbf{v}\| \geq 0$ with equality if and only if $\mathbf{v} = 0$.
- $\|\lambda\mathbf{v}\| = |\lambda|\,\|\mathbf{v}\|$
- $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

Given a norm $\|\cdot\|$ on V, show that

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$

defines a metric on V. A set U is said to be *open with respect to* $\|\cdot\|$ if it is open with respect to the associated metric d.

Solution: We first check that d is a metric. Clearly $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\| \geq 0$, and

$$\|\mathbf{v} - \mathbf{w}\| = 0 \Leftrightarrow \mathbf{v} - \mathbf{w} = 0 \Leftrightarrow \mathbf{v} = \mathbf{w}.$$

Symmetry of d follows from $\|\mathbf{v} - \mathbf{w}\| = |-1|\,\|\mathbf{w} - \mathbf{v}\|$. Finally,

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\| = \|(\mathbf{v} - \mathbf{u}) - (\mathbf{w} - \mathbf{u})\| \leq \|\mathbf{v} - \mathbf{u}\| + \|\mathbf{w} - \mathbf{u}\| = d(\mathbf{v}, \mathbf{u}) + d(\mathbf{u}, \mathbf{w}),$$

so the triangle inequality holds. Thus d is a metric.

3. Different norms $\|\cdot\|$ and $\|\cdot\|'$ on the same vector space are called *comparable* if there are constants $C_1, C_2 > 0$ such that

$$C_1 \|\mathbf{v}\| \leq \|\mathbf{v}\|' \leq C_2 \|\mathbf{v}\|$$

for all $\mathbf{v} \in V$.

Supposing that $\|\cdot\|, \|\cdot\|'$ are comparable, show that a set $U \subset V$ is open with respect to $\|\cdot\|$ if and only if it is open with respect to $\|\cdot\|'$. Does the same conclusion hold if you replace 'open' with 'closed'? 'compact'? 'connected'? Explain.

Solution: Let $U \subset V$ be open with respect to $\|\cdot\|$ and $\mathbf{v} \in U$. Then there exists $r > 0$ such that $N_r(\mathbf{v}) = \{\mathbf{w} \in V : \|\mathbf{w} - \mathbf{v}\| < r\} \subset U$. But since

$$\|\mathbf{w} - \mathbf{v}\|' \leq r/C_2 \Rightarrow \|\mathbf{w} - \mathbf{v}\| \leq r,$$

we have $N'_{r/C_2}(\mathbf{v}) \subset N_r(\mathbf{v}) \subset U$ (where the prime denotes 'neighborhood with respect to $\|\cdot\|'$. That is, any $\mathbf{v} \in U$ admits a $\|\cdot\|'$ neighborhood also contained in U, so U is open with respect to $\|\cdot\|'$.

The same argument shows that if U is open with respect to $\|\cdot\|'$, then U is also open with respect to $\|\cdot\|$. □

The conclusion also works for closed sets, compact sets, and connected sets, because all of these can be characterized in terms of open sets (e.g. a set is closed iff it's the complement of an open set, etc, etc.)

4. Let $n, m \in \mathbf{Z}^+$ be given and $V = L(\mathbf{R}^n, \mathbf{R}^m)$ be the vector space of linear transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$. Let $T = (a_{ij}) \in V$ be an arbitrary element. Show that the following norms on V are all comparable to the operator norm on V.

- $\|T\|_\infty = \max_{i,j} |a_{ij}|$
- $\|T\|_1 = \sum_{i,j} |a_{ij}|$
- $\|T\|_2 = \sqrt{\sum_{i,j} |a_{ij}|^2}$

In fact, it can be shown that pretty much any two norms on a finite dimensional vector space are comparable (Prove this and you take care of all the above items at once. And I'll give you five extra credit points).

Solution: Let $a = \max |a_{ij}|$. Then

$$a = \sqrt{a^2} \leq \sqrt{\sum_{i,j} a_{ij}^2} \leq \sqrt{\left(\sum_{i,j} |a_{ij}|\right)^2} = \sum_{i,j} |a_{ij}| \leq nma,$$

where nm is just the number of entries in T. Since all these inequalities hold regardless of T, this shows that $\|\cdot\|_\infty$, $\|\cdot\|_2$ and $\|\cdot\|_1$ are all comparable. To finish the proof it's enough to show that $\|\cdot\|$ is comparable to any one of these—say $\|\cdot\|_\infty$.

If $\mathbf{v} = \mathbf{e}_j$ is one of the usual basis vectors, then

$$\|T(\mathbf{v})\| = \|(a_{1j}, a_{2j}, \ldots, a_{mj})\| = \sqrt{\sum_i a_{ij}^2} \leq \sqrt{\sum_i a^2} = \sqrt{m}a,$$

and if $\mathbf{v} = v_1\mathbf{e}_1 + \dots v_n\mathbf{e}_n$ is an arbitrary unit vector, then

$$\|T(\mathbf{v})\| = \left\| \sum_j v_j T(\mathbf{e}_j) \right\| \leq \sum_j |v_j| \, \|T(\mathbf{e}_j)\| \leq n \cdot \sqrt{m}a$$

because $|v_j| \leq 1$ for all j. Hence

$$\|T(\mathbf{v})\| = \sup_{\|\mathbf{v}\|=1} \|T(\mathbf{v})\| \leq n\sqrt{m} \, \|T\|_\infty .$$

□

By the way,

Theorem. Any two norms on a finite dimensional real (or complex) vector space V are comparable.

Proof. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis for V and $\|\cdot\|_\infty$ be the norm on V given by

$$\|\mathbf{v}\|_\infty = \max_{1 \leq j \leq n} |c_j|$$

where the numbers c_j come from writing $\mathbf{v} = c_1\mathbf{e}_1 \dots c_n\mathbf{e}_n$ as a linear combination of basis vectors. It is enough to show that any other norm $\|\cdot\|$ on V is comparable to $\|\cdot\|_\infty$. Now on the one hand, we have

$$\|\mathbf{v}\| \leq |c_1| \, \|\mathbf{e}_1\| + \dots + \|\mathbf{e}_n\| \leq n(\max \|\mathbf{e}_j\|) \, \|\mathbf{v}\|_\infty ,$$

which gives comparability in one direction.

To get comparability in the other direction, I suppose for the sake of obtaining a contradiction that for any $C > 0$ there exists $\mathbf{v} \in V$ such that $\|\mathbf{v}\|_\infty > C \, \|\mathbf{v}\|$. Then in particular, by choosing a sequence of C's tending to ∞, we can find a sequence of vectors $\{\mathbf{v}_j\} \subset V$ such that $\|\mathbf{v}_j\|_\infty = 1$ whereas $\lim_{j\to\infty} \|\mathbf{v}_j\| = 0$.

Given this, I claim that after passing to a subsequence, we can further assume that $\{\mathbf{v}_j\}$ converges to some vector $\mathbf{v} \in V$. And I never claim anything that I can't prove. Never. If we write

$$\mathbf{v}_j = c_{1j}\mathbf{e}_1 + \dots c_{nj}\mathbf{e}_n,$$

then the 'coordinate vectors' $(c_{1j}, \dots, c_{nj}) \in \mathbf{R}^n$ all lie in the compact (because closed and bounded) set $\{(x_1, \dots, x_n) \in \mathbf{R}^n : \max |x_k| = 1\}$, so after passing to a subsequence, we can assume that $c_{1j} \to c_1$, $\dots c_{nj} \to c_n$ where $\max |c_k| = 1$. But, from the definition of $\|\cdot\|_\infty$, this is the same as saying that

$$\lim_{n\to\infty} \|\mathbf{v}_j - \mathbf{v}\|_\infty = 0$$

where $\mathbf{v} = c_1\mathbf{e}_1 + \dots + c_n\mathbf{e}_n$. So the claim is true.

We get our contradiction as follows. By the triangle inequality

$$\|\mathbf{v}_j\| - \|\mathbf{v}\| , \|\mathbf{v}\| - \|\mathbf{v}_j\| \leq \|\mathbf{v}_j - \mathbf{v}\| .$$

That is,

$$| \, \|\mathbf{v}_j\| - \|\mathbf{v}\| \, | \leq \|\mathbf{v}_j - \mathbf{v}\| \leq C \, \|\mathbf{v}_j - \mathbf{v}\|_\infty \to 0$$

as $j \to \infty$. So $\|\mathbf{v}\| = 0$. On the other hand $\mathbf{v}$ is certainly non-zero, because the basis vectors $\mathbf{e}_j$ are linearly independent and at least one of the coefficients c_j used to define $\mathbf{v}$ has magnitude 1. Since non-zero vectors must have non-zero norm, we have found our impasse and conclude that there really does exist $C > 0$ such that

$$\|\mathbf{v}\|_\infty \leq C \, \|\mathbf{v}\|$$

for every $v \in V$. □

5. Give an example of two *incomparable* norms on the (infinite dimensional) vector space $C([0,1],\mathbf{R})$ of continuous functions from $[0,1]$ to $\mathbf{R}$.

Solution: The norms

$$\|f\|_\infty := \max_{x\in[0,1]} |f(x)| \text{ and } \|f\|_1 := \int_0^1 |f(x)|\,dx$$

are incomparable. Consider for instance the functions $f_n(x) = x^n$. We have

$$\|f_n\|_\infty = |f_n(1)| = 1$$

for every $n \in \mathbf{N}$, but

$$\|f_n\|_1 = \frac{1}{n+1} \to 0.$$

Hence, there is no constant $C > 0$ such that

$$\|f\|_\infty \leq C\,\|f\|_1$$

for all $f \in C([0,1],\mathbf{R})$.