Solutions to Homework 2

Rudin, page 239/6: Since f is a rational function whose denominator vanishes only at the origin, it is clear that the partial derivatives of f exist and are continuous everywhere except $(x, y) = (0, 0)$. Now at the origin, we have

$$D_1 f(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h \cdot 0 - 0}{h^2 + 0} == \lim_{h\to 0} \frac{0}{h^2} = 0.$$

A similar computation shows that $D_2 f(0,0) = 0$.

However, if we consider the sequence of points $p_n = (1/n, 1/n)$, then

$$\lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} \frac{1/n^2}{2/n^2} = \frac{1}{2} \neq 0 = f(0,0) = f(\lim_{n\to\infty} p_n).$$

So f is not continuous at $(0, 0)$. □

Rudin, page 239/7: Let $p = (x_1, \dots, x_n), q = (y_1, \dots, y_n) \in E$ be any two points. For $k = 1, \dots, n$ let $p_k = (x_1, \dots, x_k, y_{k+1}, \dots, y_n)$ (in particular $p_n = p$), and set $p_0 := 0$. Then for $1 \le k \le n$, the points p_k and p_{k-1} differ only in the kth coordinate. Hence, by the one-variable mean value theorem, there exists c_k between x_k and y_k such that

$$|f(p_k) - f(p_{k-1})| = |D_k f(c_k)(x_k - y_k)| \le C \|x_k - y_k\|,$$

where $C > 0$ is an upper bound for $D_1 f, \dots D_n f$ on E. Hence

$$|f(p) - f(q)| \le \sum_{k=1}^{n} |f(p_k) - f(p_{k-1})| \le C \sum_{k=1}^{n} |x_k - y_k| \le Cn \|p - q\|.$$

The main thing is that the constant Cn has nothing to do with p or q.

Now let $p \in E$ be any point and $\{p_n\} \subset E$ be any sequence convergin to p. Then

$$0 \le \lim_{n\to\infty} \|f(p_n) - f(p)\| \le C \lim_{n\to\infty} \|p_n - p\| = 0.$$

In other words $\lim_{n\to\infty} f(p_n) = f(p)$, which shows that f is continuous at p. Since p was arbitrary, we conclude that f is continuous on E. □

Rudin, page 239/8: Since f is differentiable at x, all partial derivatives $Df_1, \dots Df_n$ exist at x and $Df(x) = (Df_1(x), \dots, Df_n(x))$. And if f has a local maximum at the point $x = (x_1, \dots, x_n)$, then for any $1 \le k \le n$, the one-variable function

$$g_k(t) := f(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n)$$

has a local maximum at $t = x_k$. In particular $g_k'(x_k) = 0$. But $g_k'(x_k)$ is just $D_k f(x)$. Hence $Df(x) = (0, \dots, 0)$. □

Rudin, page 239/9: Fix any point $p \in E$. Let

$$K = \{x \in E : f(x) = f(p)\}.$$

I will show that $K = E$, which implies of course that f is constant. Now $p \in K$, so $K \neq \emptyset$. Moreover, f is continuous because it is differentiable, so $K = f^{-1}(f(p))$ is closed (i.e. the inverse image of a closed set by a continuous function is closed). In particular, $E - K$ is open. If I can show that K is also open, then

$$E = K \cup (K - E)$$

will express E as a disjoint union of two open sets. But E is connected by hypothesis, so it will follow that $K - E =$ and therefore $E = K$.

So to summarize, it's enough to show that K is open. Let $x \in K$ be any point. Because E itself is open, we can choose $r > 0$ such that $N_x(r) \subset E$. Let $y \in N_x(r)$ be any other point and

$$h : [0,1] \to \mathbf{R}$$

be given by $h(t) = f(tx + (1-t)y)$. Then $h(1) = f(x) = f(p)$ and $h(0) = f(y)$. Moreover, as the composition of two differentiable functions h itself is differentiable, with

$$h'(t) = Df(tx + (1-t)y) \cdot \frac{d}{dt}(tx + (1-t)y) = \mathbf{0} \cdot (x - y) = 0$$

for all $t \in [0,1]$. It follows (from one variable calculus) that h is constant. In particular $f(y) = h(1) = h(0) = f(p)$. And $y \in N_r(x)$ was arbitrary, so we conclude that $N_r(x) \subset K$. As x was arbitrary, too, it follows that K is open. □

Rudin, page 239/14:

a) For $(x,y) \neq (0,0)$, a quick computation shows that

$$D_1 f(x,y) = \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2}, \quad D_2 f(x,y) = \frac{-2x^2y}{(x^2+y^2)^2}.$$

Taking $D_1(f(x,y))$, for example, we note that both x^4 and x^2y^2 are smaller than $\|(x,y)\|^2$. Hence

$$|D_1 f(x,y)| \leq \frac{4\,\|(x,y)\|^2}{\|(x,y)\|^2} = 4$$

for all $(x,y) \in \mathbf{R}^2 - (0,0)$. Similarly, $|D_2 f| \leq 2$ on $\mathbf{R}^2 - (0,0)$. Finally, for $(x,y) = (0,0)$ one computes

$$D_1 f(0,0) = \lim_{h \to 0} \frac{(h-0)}{h} = 1$$

and, in the same fashion, $D_2 f(0,0) = 0$.

b) Let us write $\mathbf{u} = (s, t)$. Then

$$\begin{aligned} D_{\mathbf{u}} f(0,0) &= \lim_{h \to 0} \frac{f(h\mathbf{u}) - f(\mathbf{0})}{h} \\ &= \lim_{h \to 0} \frac{h^3 s^3 / (h^2 s^2 + h^2 t^2)}{h} = \frac{s^3}{s^2 + t^2}, \end{aligned}$$

which shows that $D_{\mathbf{u}} f(0,0)$ exists for any $\mathbf{u} \neq \mathbf{0}$.

d) Were f actually differentiable at $(0,0)$, $D_{\mathbf{u}} f$ would be linear in $\mathbf{u}$. But from the formula computed in part b), we see that

$$D_{(0,1)} f(0,0) + D_{(1,0)} f(0,0) = 0 + 1 \neq \frac{1}{2} = D_{(1,1)} f(0,0).$$