Solutions to Homework 3

Rudin, page 239/9: Fix any point $p \in E$. Let

$$K = \{x \in E : f(x) = f(p)\}.$$

I will show that $K = E$, which implies of course that f is constant. Now $p \in K$, so $K \neq \emptyset$. Moreover, f is continuous because it is differentiable, so $K = f^{-1}(f(p))$ is closed (i.e. the inverse image of a closed set by a continuous function is closed). In particular, $E - K$ is open. If I can show that K is also open, then

$$E = K \cup (K - E)$$

will express E as a disjoint union of two open sets. But E is connected by hypothesis, so it will follow that $K - E =$ and therefore $E = K$.

So to summarize, it's enough to show that K is open. Let $x \in K$ be any point. Because E itself is open, we can choose $r > 0$ such that $N_x(r) \subset E$. Let $y \in N_x(r)$ be any other point and

$$h : [0, 1] \to \mathbf{R}$$

be given by $h(t) = f(tx + (1 - t)y)$. Then $h(1) = f(x) = f(p)$ and $h(0) = f(y)$. Moreover, as the composition of two differentiable functions h itself is differentiable, with

$$h'(t) = Df(tx + (1 - t)y) \cdot \frac{d}{dt}(tx + (1 - t)y) = \mathbf{0} \cdot (x - y) = 0$$

for all $t \in [0, 1]$. It follows (from one variable calculus) that h is constant. In particular $f(y) = h(1) = h(0) = f(p)$. And $y \in N_r(x)$ was arbitrary, so we conclude that $N_r(x) \subset K$. As x was arbitrary, too, it follows that K is open. □

Rudin, page 239/14:

a) For $(x, y) \neq (0, 0)$, a quick computation shows that

$$D_1 f(x, y) = \frac{x^4 + 3x^2y^2}{(x^2 + y^2)^2}, \quad D_2 f(x, y) = \frac{-2x^3 y}{(x^2 + y^2)^2}.$$

Taking $D_1(f(x, y))$, for example, we note that both x^4 and x^2y^2 are smaller than $|(x, y)|^4$. Hence

$$|D_1 f(x, y)| \leq \frac{4|(x, y)|^2}{|(x, y)|^2} = 4$$

for all $(x, y) \in \mathbf{R}^2 - (0, 0)$. Likewise,

$$|D_2 f(x, y)| \leq \frac{2|(x, y)|^4}{|(x, y)|^4} = 2.$$

Finally, for $(x, y) = (0, 0)$ one computes

$$D_1 f(0,0) = \lim_{h \to 0} \frac{(h-0)}{h} = 1$$

and, in the same fashion, $D_2 f(0,0) = 0$.

b) Let us write $\mathbf{u} = (s, t)$. Then

$$\begin{aligned} D_{\mathbf{u}} f(0,0) &= \lim_{h \to 0} \frac{f(h\mathbf{u}) - f(\mathbf{0})}{h} \\ &= \lim_{h \to 0} \frac{h^3 s^3/(h^2 s^2 + h^2 t^2)}{h} = \frac{s^3}{s^2 + t^2} = s^3, \end{aligned}$$

since $\mathbf{u}$ is a *unit* vector. This shows that $D_{\mathbf{u}} f(0,0)$ exists and, moreover, since $|s| \leq |\mathbf{u}| \leq 1$, we have $|D_{\mathbf{u}} f(0,0)| \leq 1$.

d) Were f actually differentiable at $(0,0)$, $D_{\mathbf{u}} f$ would be linear in $\mathbf{u}$. But from the formula computed in part b), we see that

$$D_{(0,1)} f(0,0) + D_{(1,0)} f(0,0) = 0 + 1 \neq \frac{1}{2} = D_{(1,1)} f(0,0).$$

Rudin, page 239/16: From the definition of derivative, we have

$$f'(0) = \lim_{h \to 0} \frac{h + 2h^2 \sin(1/h) - 0}{h} = 1 + \lim_{h \to 0} 2h \sin(1/h) = 1$$

since $|2h \sin(1/h)| \leq 2|h| \to 0$ as $h \to 0$. In particular, $f'(0)$ is invertible (i.e. non-zero).

Moreover, for $t \neq 0$, we have

$$f'(t) = 1 + 4t \sin(1/t) - 2\cos(1/t).$$

Hence $|f'(t)| \leq 1 + 4|t| + 2 < 7$ for $t \in (-1, 1)$. So f' is bounded on $(-1, 1)$.

Now suppose that f is actually injective on some neighborhood $I = (-\epsilon, \epsilon)$ of 0. Then because f is continuous, it follows that f is actually strictly monotone—say for the moment that f is strictly increasing. Then at any point $x \in I$, we have

$$f'(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \geq 0$$

because $h > 0$ implies that $x + h > x$ which implies in turn that $f(x+h) > f(x)$.

But in fact f' is not non-negative on I: at any point $x = 1/2n\pi$ we have $f'(x) = 1 + 4 \cdot (1/2n\pi) \cdot 0 - 2 \cdot 1 = -1$. It follows that f cannot be strictly increasing on I.

So it must be that f is strictly *decreasing* on I. As before we conclude that $f'(x) \leq 0$ for every $x \in I$. This contradicts the fact that $f'(0) = 1$, though. So f is not strictly decreasing, therefore not monotone, and therefore not injective on I. Too bad. □

Rudin, page 239/17:

a) The range is all $\mathbf{R}^2$ except $(0,0)$. Given $(s,t) \in \mathbf{R}^2$, just let $x = \frac{1}{2}\log(s^2+t^2)$ and choose y so that $\cos y = e^{-x}s$ and $\sin y = e^{-x}t$. This can be done since $(e^{-x}s)^2 + (e^{-x}t)^2) = 1$.

b) The Jacobian of f is

$$\det \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} \neq 0$$

for any $(x,y) \in \mathbf{R}^2$. So f is locally invertible by the inverse function theorem. However, f is not globally invertible since it's not injective: $f(x, y+2n\pi) = f(x,y)$ for every $n \in \mathbf{N}$.

c)

$$g(s,t) = (\frac{1}{2}\log(s^2+t^2), \tan^{-1}(t/s)),$$

(choosing $\tan^{-1}$ to have range $(-\pi/2, \pi/2)$). Then

$$g'(s,t) = \begin{pmatrix} \frac{s}{s^2+t^2} & \frac{t}{s^2+t^2} \\ \frac{-t}{s^2+t^2} & \frac{s}{s^2+t^2} \end{pmatrix}.$$

So

$$g'(f(x,y)) = \begin{pmatrix} e^{-x}\cos y & e^{-x}\sin y \\ -e^{-x}\sin y & e^{-x}\cos y \end{pmatrix} = f'(x)^{-1}.$$

d) For given y, the set

$$\{f(x,y) : x \in \mathbf{R}\} = \{e^x(\cos y, \sin y) : x \in \mathbf{R}\}$$

consists of all positive multiples of the fixed vector $(\cos y, \sin y)$. That is, the image of a horizontal line is a ray beginning at $(0,0)$ (but not including this point).

For given x, the set

$$\{f(x,y) : y \in \mathbf{R}\} = \{e^x(\cos y, \sin y) : y \in \mathbf{R}\}$$

consists of all points at distance e^x from $(0,0)$. That is, the image of a vertical line is a circle centered at the origin.

Supplementary problem 1: We have

$$f'(x,y) = \begin{pmatrix} \cos x & \sin y \\ e^x & e^y \end{pmatrix}$$

which clearly varies continuously with (x,y). So f is a C^1 function on all of $\mathbf{R}^2$.

- We check

$$\det f'(0,0) = \det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1.$$

In particular $f'(0,0)$ is invertible, and the inverse function theorem implies that f is locally invertible at $(0,0)$. The inverse function has linear approximation at $(-1,2)$ given by

$$\begin{aligned} f^{-1}(-1,2)+(f^{-1})'(-1,2)\begin{pmatrix} x+1 \\ y-2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + f'(0,0)^{-1}\begin{pmatrix} x+1 \\ y-2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x+1 \\ y-2 \end{pmatrix} \\ &= \begin{pmatrix} x+1 \\ y-x-3 \end{pmatrix} \end{aligned}$$

- Let $(x_0, y_0) = (0,0)$ be a first guess at the desired point (x,y), and let us use Newton's method (and Mathematica!)

$$(x_{n+1}, y_{n+1}) = (x_n, y_n) - f'(x_n, y_n)^{-1}(f(x_n, y_n) - (-1.02, 1.97))$$

to improve our guesses til they settle down to three unchanging decimal places.

Thus,

$$\begin{aligned} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1.02 \\ 1.97 \end{pmatrix}\right) \\ &= -\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} .02 \\ .03 \end{pmatrix} = \begin{pmatrix} -.02 \\ -.01 \end{pmatrix} \end{aligned}$$

And again

$$\begin{aligned} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} -.02 \\ -.01 \end{pmatrix} - \begin{pmatrix} .9998 & -.0100 \\ .9802 & .9900 \end{pmatrix}^{-1}\left(\begin{pmatrix} -1.01995 \\ 1.970255 \end{pmatrix} - \begin{pmatrix} -1.02 \\ 1.97 \end{pmatrix}\right) \\ &= \begin{pmatrix} -.0200533 \\ -.0101982 \end{pmatrix} \end{aligned}$$

A further iteration of Newton's method shows that (x_3, y_3) is exactly the same as (x_2, y_2) to the number of digits shown above, so (x_2, y_2) ought to be approximate (x, y) accurately to at least 5 (non-zero) digits.