Solutions to Homework 5

Supplementary problem 1: Since C is defined implicitly as the 0-level set of a function $f : \mathbf{R}^3 \to \mathbf{R}$, we will have that C is a codimension one (i.e. dimension 2) C^1 submanifold of $\mathbf{R}^3$ at all points where f' has maximal rank 1. Specifically,

$$f'(x, y, z) = (2x, -2y, -2z)$$

has rank one unless all three entries vanish—i.e. unless $(x, y, z) = (0, 0, 0)$ So C is a two dimensional submanifold of $\mathbf{R}^3$ except at the origin (which, incidentally, actually does lie in C).

The picture you draw should show a circular cone generated by rotating the line $x = y$ about the x axis.

Supplementary problem 2: The function $h : \mathbf{R} \to \mathbf{R}^2$ is clearly C^1 and parametrizes the set C. Therefore C is a dimension one (i.e. codimension 1) submanifold of $\mathbf{R}^2$ except at points where h' fails to have (maximal) rank one. Specificly,

$$h'(t) = (2t, 3t^2),$$

which has rank equal to one unless both entries vanish–i.e. unless $t = 0$. Hence C is a dimension one submanifold at all points other than $h(0) = (0, 0)$.

The tangent space to C at $h(t)$ is the image of $\mathbf{R}$ under the linear transformation $h'(t)$. In particular, since '1' is a basis for $\mathbf{R}$, the vector $h'(t) \cdot 1 = (2t, 3t^2)$ will be a basic for the tangent space to C at $h(t)$.

Supplementary problem 3: The set C is just the $(1, 0)$ level set of the function $f : \mathbf{R}^2 \to \mathbf{R}^4$. Therefore C will be a submanifold of $\mathbf{R}^4$ with both dimension and codimension equal to two except at points where

$$f'(x, y, z, w) = \begin{pmatrix} 2x & -2y & -w & -z \\ w & z & y & x \end{pmatrix}$$

has rank less than two. Having rank less than two means that *all* 2×2 submatrices of f' have zero determinant. There are six of these, and their determinants are

$$2xz + 2yw, \quad 2xy + w^2, \quad 2x^2 + wz, \quad -2y^2 + wz, \quad -2xy + z^2, \quad -xw + yz.$$

If all six vanish, then comparing the second and fourth expression shows that $w^2 = -z^2$, which can only happen if $z = w = 0$; and the third and fifth combine to imply that $x^2 = -y^2$, so that again $x = y = 0$. Hence the only trouble occurs at $(0, 0, 0, 0)$ which is not actually in C. So C is a submanifold of $\mathbf{R}^4$ at all points.

Now the tangent space to C at $(1, 1, 0, 0)$ is just the $(1, 0)$ level set of the linear approximation of f at that point. That is, we seek all vectors (a, b, c, d) satisfying

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + f'(1, 1, 0, 0) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

From this it is clear that we're just after the nullspace of

$$f'(1,1,0,0) = \begin{pmatrix} 2 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

After dividing the matrix by two, it's in reduced row echelon form with free variables b and d. The nullspace is therefore the two dimensional subspace of $\mathbf{R}^4$ generated by the vectors $(1,1,0,0)$ and $(0,0,-1,1)$ obtained by setting one free variable equal to one and the other equal to 0.

Supplementary problem 4: We have

$$f'(x,y) = \begin{pmatrix} 3x^2 - 4xy & -2x^2 + 1 \\ -4x & 3y^2 \end{pmatrix}$$

With a first guess of $(x_0, y_0) = (1,1)$, I find a second guess (x_1, y_1) by linearly approximating f about $(1,1)$ and setting the result equal to $(1.1, -.8)$.

$$\begin{pmatrix} .1 \\ -.8 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 - 1 \\ y_1 - 1 \end{pmatrix}$$

which implies that

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ -4 & 3 \end{pmatrix}^{-1} \begin{pmatrix} .1 \\ .2 \end{pmatrix} = \begin{pmatrix} .92857 \\ .97143 \end{pmatrix}$$

To find a better guess it is necessary to repeat this process using the linear approximation of f near (x_1, y_1) instead of (x_0, y_0). Rather than type up the details, I have made available the mathematica worksheet (web viewable version available, too) that I used to do this computation. See the course webpage for the link.