Solutions to Homework 6

Rudin, page 165/20:
Proof. Let $P(x) = c_n x^n + \ldots + c_1 x + c_0$ be a polynomial. Then

$$\int_0^1 f(x)P(x)\,dx = c_n \int_0^1 f(x)x^n\,dx + \ldots c_1 \int_0^1 f(x)x\,dx + c_0 \int_0^1 f(x)x^0\,dx = 0$$

by hypothesis. By Weierstrass' Approximation Theorem, there is a sequence $\{P_n\}$ of polynomials converging uniformly to f on $[0,1]$. Moreover, f is continuous and therefore bounded on $[0,1]$, as is each of the polynomials P_n. Therefore (see problem 2 from this section), $f \cdot P_n$ converges uniformly to $f \cdot f = f^2$ on $[0,1]$. By Theorem 7.16, I conclude that

$$0 = \lim_{n\to\infty} \int_0^1 f(x)P_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f(x)P_n(x)\,dx = \int_0^1 [f(x)]^2\,dx.$$

Now f^2 is a non-negative function that vanishes exactly where f does. So if $f(x) > 0$ for some $x \in [0,1]$, it follows from continuity of f that for some $\delta > 0$, $|t - x| < \delta$ implies that $f(t) > f(x)/2$. Therefore,

$$\int_0^1 [f(x)]^2\,dx \geq \delta[f(x)]^2/4 > 0.$$

This is impossible, so $f \equiv 0$ on $[0,1]$. □

Rudin, page 165/21: The identify function $e^{i\theta} \mapsto e^{i\theta}$ belongs to $\mathcal{A}$, vanishes nowhere and is injective. Hence $\mathcal{A}$ is nowhere vanishing and separates points. Nevertheless, I claim that the function $f(e^{i\theta}) = 1/e^{i\theta} = e^{-i\theta}$ is not in the uniform closure of $\mathcal{A}$.
Proof. Note first that

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\,d\theta = 2\pi.$$

Now suppose in order to obtain a contradiction that f is in the uniform closure of $\mathcal{A}$. Then for any $\epsilon > 0$ we could find an element $g \in \mathcal{A}$ such that

$$|g(e^{i\theta}) - f(e^{i\theta})| < \epsilon$$

for every $e^{i\theta}$.

$$\left| \int_0^{2\pi} g(e^{i\theta})e^{i\theta}\,d\theta - \int_0^{2\pi} f(e^{i\theta})e^{i\theta}\,d\theta \right| \leq \int_0^{2\pi} |g(e^{i\theta}) - f(e^{i\theta})||e^{i\theta}|\,d\theta \leq 2\pi\epsilon < 2\pi.$$

provided we choose $\epsilon < 1$. In particular,

$$\int_0^{2\pi} g(e^{i\theta})e^{i\theta} \neq 0.$$

On the other hand $g(\theta) = \sum_{n=0}^{N} c_n e^{in\theta}$, so

$$\int_0^{2\pi} g(e^{i\theta})\,d\theta = \sum_{n=0}^{N} c_n \int_0^{2\pi} e^{in+1\theta}\,d\theta = \sum_{k=1}^{N+1} c_{k-1}\left(\int_0^{2\pi} cos(k\theta)\,d\theta + i\int_0^{2\pi} sin(k\theta)\,d\theta\right) = 0.$$

This contradicts the above and proves that f is not in the uniform closure of $\mathcal{A}$. □

Rudin, page 165/22:
Proof. Let $\epsilon > 0$ be given. By Exercise 12 from Chapter 6, there is a continuous function $h : [a,b] \to \mathbf{R}$ such that $\|h - f\|_2 < \sqrt{\epsilon/4}$. In other words,

$$\int_a^b |h(x) - f(x)|^2\,dx < \epsilon/4.$$

By Weiestrass' Approximation Theorem, there is a polynomial P such that

$$|P(x) - h(x)| < \sqrt{\frac{\epsilon}{4(b-a)}}$$

for all $x \in [a,b]$. Thus

$$\int_a^b |P(x) - h(x)|^2\,dx < (b-a)\frac{\epsilon}{4(b-a)} = \epsilon/4.$$

Finally,

$$|P(x) - f(x)|^2 \le (|P(x) - h(x)| + |h(x) - f(x)|)^2 \le 2(|P(x) - h(x)|^2 + |h(x) - f(x)|^2),$$

so

$$\int_a^b |P(x) - f(x)|^2\,dx < 2(\epsilon/4 + \epsilon/4) = \epsilon.$$

If I now choose a sequence $\epsilon_n > 0$ tending to 0 and let P_n be the corresponding polynomials, then it follows that

$$\lim_{n\to\infty} \int_a^b |P_n(x) - f(x)|^2\,dx = 0$$

□

Rudin, page 165/23:
Proof. I proceed by induction to show that for all $n \ge 0$ and $|x| < 1$ that

$$0 \le P_n(x) \le P_{n+1}(x) \le |x|$$

and

$$|x| - P_n(x) \le |x|\left(1 - \frac{|x|}{2}\right)^n.$$

For the moment, let me suppose that these inequalities are proven. By finding roots of the derivative, it is easily shown that the function $h : [0,1] \to \mathbf{R}$ given by $h(t) = t(1-t/2)^n$ will achieve its maximum at $t = 0, 1$ or $2/(n+1)$ (i.e. at endpoints or critical points). We have $h(0) = 0$, $h(1) = 1/2^n$, and $h(2/n+1) < 2/(n+1)$ (since $h(x) < x$ when $x \in (0,1]$). In any case, $h(t) < 2/(n+1)$ for all t. Thus

$$||x| - P_n(x)| = |x| - P_n(x) \leq h(x) < 2/(n+1)$$

for all $x \in [-1,1]$, and it follows that P_n converges uniformly to $|x|$.

Now I return to the proof of the inequalities asserted earlier. When $n = 0$, we have $P_n(x) = 0$ and $P_{n+1}(x) = x^2/2$, and all the inequalities are easily verified directly. So now I assume that the inequalities have been verified for $n = k$, and I will prove that they hold when $n = k+1$. First of all, we use $0 \leq P_k(x) \leq |x|$ to estimate

$$P_{k+1}(x) = P_k(x) + \frac{x^2 - P_k^2(x)}{2} \geq P_k(x) + \frac{x^2 - |x|^2}{2} = P_k(x).$$

Secondly,

$$|x| - P_{k+1}(x) = [|x| - P_k(x)]\left[1 - \frac{|x| + P_k(x)}{2}\right] \geq [|x| - P_k(x)]\left[1 - \frac{|x| + |x|}{2}\right] \geq 0$$

for $|x| \leq 1$. So $P_{k+1}(x) \leq |x|$. Finally, in the other direction

$$\begin{aligned} |x| - P_{k+1}(x) &= [|x| - P_k(x)]\left[1 - \frac{|x| + P_k(x)}{2}\right] \\ &\leq [|x| - P_k(x)]\left[1 - \frac{|x|}{2}\right] \\ &\leq |x|\left(1 - \frac{|x|}{2}\right)^k \left[1 - \frac{|x|}{2}\right] \\ &= |x|\left(1 - \frac{|x|}{2}\right)^{k+1}, \end{aligned}$$

where the second inequality comes from the induction hypothesis. This completes the induction step and the proof. □