

Final for Math 405, Introduction to Combinatorics.

Name: _____

December 12, 2000

Instructions: The test will be 120 minutes in length.

1. (10 pts) Calculate the (multinomial) coefficient of the monomial a^4b^3c in the expansion of $(a + b + c)^8$:

2. (10 pts) Give a detailed proof why the formula

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0.$$

holds for all positive integers n .

3. a) (7 pts) Find the number of nonnegative integral solutions of the equation

$$a + b + c + d = 100$$

a) (8 pts) Same as a) with the additional requirement that $a \geq 10$, a is even and $0 \leq b \leq 1$.

4. (10 pts) Recall that the Euler ϕ function $\phi(n)$ is defined as the number of integers $1 \leq m < n$ such that $\gcd(m, n) = 1$. Compute (Hint use inclusion/exclusion principle):

a) (5 pts) $\phi(101) =$

b) (5 pts) $\phi(143) =$

c) (5 pts) $\phi(1001) =$

5. (10 pts) Assume that

$$h_n = 3h_{n-1} - 2h_{n-2}$$

for $n \geq 3$ and that $h_1 = h_2 = 1$. Compute h_n for all integers $n \geq 3$.

6. (10 pts) Solve explicitly the recurrence relation

$$\begin{aligned}h_{n+1} &= 2h_n + 7, \quad n \geq 1. \\h_1 &= 2.\end{aligned}$$

7. (15 pts) Let

$$f(x) = \frac{1}{x(x-1)(x-2)(x-3)} = \sum_{n=-1}^{\infty} s_n x^n$$

be the generating function of a sequence. Find an explicit formula for s_n . (Hint: Think what the effect of the factor $\frac{1}{x}$ might be.)

8. (15 pts) Determine the number of n digit numbers with all digits odd, such that 1 and 3 each occur an even number of times.

9. (10 pts) Consider the binary $[7, 4]$ code having parity check matrix

$$H := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Assume one error did happen and the vector $y = (0, 1, 1, 0, 0, 0, 0)$ was received. Find the code word which was sent.

10. (10 pts) Let $\mathbb{F} := \{0, 1, 2\}$ be the field of three elements. Count the number of 2-dimensional subspaces in \mathbb{F}^4 .

11. (15 pts) Let $\mathbb{F} := \mathbb{F}_{11} = GF(11)$ be the Galois field of 11 elements. Construct an ISBN-code $C \subset \mathbb{F}^{10}$ which contains more than 200'000'000 code-words (ISBN-numbers) and which is capable of correcting one error. (Note that the ISBN system in use only can detect an error and there is no way of correcting an error).

12. (15 pts) Alice publishes the encryption function

$$\varphi : \mathbb{Z}_{33} \rightarrow \mathbb{Z}_{33}, \quad x \mapsto x^7.$$

Bob, not aware that everybody can compute the factorization of $n = 33$ sent to Alice the message $17 = x^7$. Find the message x of Bob. Note: You should not use trial and error and try out all 32 possibilities.