Information on Mathematics 421
Algebraic Geometry, Fall 1996

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Contents

General information

This course is new this year and therefore the material will be adjusted somewhat for next year.

The enrollment is currently stable at 20. The maximum enrollment was 23 or 24. All students are mathematics majors and all but one of the students are seniors. Overall the students are quite reasonable and I think that they are enjoying the class (as am I).

I give a homework assignment every week. Usually four problems: I have appended the problems given so far this year. I have broken the class up into collaborative learning groups of three and four students. Each group works as a group on the homework, which I grade and return. I also put answers to the homework on the network. Besides the homework, which is worth 100 points, I give two tests worth 100 points each and a final worth 150 points. I have appended a copy of the first test and my handout with basic information.

I plan to use the computer more in the class next year. I often suggest using maple for assignments, and I have begun giving demonstrations in class.

Besides the three classes on Monday, Wednesday, and Friday, I meet my class every Tuesday from 7:15 PM to 8:15 PM to answer questions, review and fill in gaps in people's backgrounds, and go over problems on the homework or on tests that caused difficulties.

We have already covered the real and complex projective line and plane. We have covered the homogeneous form of the tangent line to a curve, Euler's theorem on homogeneous functions, and how to find the singular points of plane curves. We have covered Bézout's theorem (with the proof so far only for two curves one of which is a line or a conic, though as a consequence of the basic theory of resultants which developed in class have shown how to find all the solutions of a system of two polynomials in two unknowns over the complex numbers). We spent quite some time going over complex numbers, which many students had (to me shockingly) never seen. I went over calculus in the plane, i.e., power series, the Cauchy-Riemann equations, use of Green's theorem in the plane, i.e., the Cauchy integral theorem. I also have covered divisors of rational functions and one forms on the line, and covered calculation of the Euler characteristic of Riemann surfaces. My most hopeful plan was to study the projective plane and algebraic curves, leading up to a proof of special cases of Riemann-Roch for curves and applications of it. Some of this was too ambitious, and I have moderated my plans. Also a good number of the students have forgotten a fair amount of the algebra and calculus that they learned. I shifted to covering very concrete topics, e.g., proof of partial fraction decomposition; the proof of basic results about resultants and discriminants (including the proof that polynomial rings over fields are unique factorization domains) leading up to a constructive proof of Bézout's theorem. This has been going very well. I will then define affine varieties and their functions using polynomial rings. I will explain the Nullstellensatz and set up the usual dictionary between affine varieties and quotient rings of the ring of polynomials. I will if possible prove the Nullstellensatz. I will then cover the basic results on computational algebraic geometry, i.e., Groebner bases. This is very down to earth, and the students like the concrete algebra. I will try to cover the basic theory of symmetric functions and some theory of the symmetric group, which fits in nicely with the above.

The book by Reid turned out to be unsuitable for the class. Besides being a little too hard, the material in any part was too dependent on material already covered. I wrote some notes for some topics covered in the class (they are on the network), but would still like them to have a book I can assign reading in. I have followed Walker's *Algebraic Curves*, which I put on reserve in the library, for my presentation of Unique factorization domains, resultants, discriminants. I have done this carefully and gone over it enough that it is understood. Next year I plan to use the book *Ideals, Varieties, and Algorithms* by Cox, Little, and O'Shea, which was out of print (it was between editions) at the the time I had to order a course book.

1 Syllabus for Next Year

For next year my plan would be to

1. spend the first few weeks going over complex numbers and enough calculus in the plane to prove the fundamental theorem of algebra;
2. develop the basic theory about the projective line and plane over the real and complex numbers and algebraic plane curves, especially those of degree up to three;
3. introduce tangent lines and singular points;
4. study polynomials and rational functions on the line and plane (including partial fraction expansions, one and two-forms, linear fractional transformations, and basic elimination theory);
5. define affine varieties and their functions using polynomial rings;
6. explain the Nullstellensatz and set up the dictionary between affine varieties and quotients of the ring of polynomials (and if possible prove the Nullstellensatz);
7. cover the basic results on computational algebraic geometry, i.e., Groebner bases;
8. cover the basic theory of symmetric functions and some theory of the symmetric group.

1.1 Requirements for the Course

The requirements should be increased. Besides the calculus sequence Mathematics 125 to 226; and the courses on algebraic structures and linear algebra; I would like to add real analysis or ordinary differential equations.

2 Class Handout

Instructor: Andrew Sommese
231 CCMB (On Juniper, just south of the main library)
Phone: 631-6498; e-mail: sommese.1@nd.edu
Text: *Undergraduate Algebraic Geometry*, by M. Reid

I am in my office almost all of every weekday, and encourage students to visit any time. If you just come to my office you will probably find me, but if you set up a time with me before hand, then you can be sure I will be there.

Examinations, homework and grades There will be two departmental examinations and one final examination (whose dates and times are listed below). Each departmental exam is a one-hour exam and will be worth 100 points. The final exam is a two-hour exam and will be worth 150 points. Homework will be worth 100 points. The final exam will cover all the material of the course. The total number of possible points for the semester is 450. The numerical break points for letter grades (A, A-, B+,...) will be based only on the test scores and the homework.

Homework will be assigned regularly, and is an integral part of the course. I ask students to form groups of three (and one or two groups of size four depending on the number of students in the class mod 3) to do all the assignments. If there are people who would like to be in the same group, please let me know by the end of class on Friday, August 30. I will hand out the lists of members of the class by groups on Monday, September 2. If the class enrollment changes I might have to add members to a few groups. Typically I will give the week's assignment on a Monday and collect it the following Monday. I strongly encourage students to see me if there is anything they are unclear on would like to know more about.

Both examinations and the homework are conducted under the honor code. People within a group are graded together on the homework assignments, and are expected to work together. People in different groups are encouraged to discuss the mathematics, but should not discuss how to do the week's assignment before it is handed in!

A student who misses an examination will receive zero points for that exam unless he or she has written permission from the Vice President for Student Affairs.

Exam Dates

Exam 1:	Wednesday, October 2, 1996 (in class)
Exam 2:	Friday, November 15, 1996 (in class)
Final:	Thursday, December 19, 1996, 8–10AM
	(Location of final exam will be announced later.)

3 Homework

Generally there has been one week give for the completion of each assignment.

Problems due September 9

Problem 1 (2 points) *For an inscribed n-sided regular polygon in the unit circle, show that the the product of the lengths of the $n-1$ segments from one fixed vertex to the remaining vertices equals n for all $n \geq 2$.*

Problem 2 (5 points) *Find the singular points of the curve $zxy - z^3 = 0$. Write the equation and sketch the graph in the coordinates for the standard Euclidean $\mathbb{R}^2$ centered at $[0,0,1]$ and also at $[1,0,0]$.*

Problem 3 (4 points) *Show that the Klein quartic, $x^3y + y^3z + z^3x = 0$, is smooth. Show also that it is invariant under the projectivity $[x,y,z] \to [y,z,x]$ and also under the projectivity $[x,y,z] \to [x, \alpha^2 y, \alpha^3 z]$ where α is a 7-th root of unity, i.e., $\alpha^7 = 1$.*

Problems due September 18

Problem 4 (3 points) *Find all automorphisms of $\mathbb{P}^1_{\mathbb{C}}$, $f : [z,w] \to [az + bw, cz + dw]$, such that $f(f([z,w])) = [z,w]$. Find all automorphisms of $\mathbb{P}^1_{\mathbb{C}}$, $f : [z,w] \to [az + bw, cz + dw]$ such that, in addition to $f(f([z,w])) = [z,w]$, $f([1,1]) = [1,1]$ and $f([2,1]) = [2,1]$. (Find all means in particular to show the automorphisms that you find are all of the automorphisms.)*

Problem 5 (2 points) *Find all automorphisms of $\mathbb{P}^1_{\mathbb{C}}$, $f : [z,w] \to [az + bw, cz + dw]$, such that $f([0,1]) = [0,1]$, $f([1,1]) = [1,1]$, and $f([1,0]) = [1,0]$. (Find all means in particular to show the automorphisms that you find are all of the automorphisms.)*

Problem 6 (4 points) *Write down the homogeneous degree one polynomial that defines the tangent line in $\mathbb{P}^2_{\mathbb{C}}$ to the curve C defined by $x^{d-1}y + y^{d-1}z + z^{d-1}x = 0$ at $[0,0,1]$. What points does this line meet C? What are their multiplicities.*

Problems due September 25

Problem 7 (3 points) *We have a natural group homomorphism from $\mathrm{SL}(2,\mathbb{C})$, the 2×2 complex matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant 1, to the group $\mathrm{PL}(1,\mathbb{C})$ of automorphisms, $[z,w] \to [az + bw, cz + dw]$, of $\mathbb{P}^1_{\mathbb{C}}$. What is the kernel of this mapping? Show that the mapping $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{PL}(1,\mathbb{C})$ is onto.*

Problem 8 (3 points) *Find the points in $\mathbb{P}^2_{\mathbb{C}}$ on the curves $x - y = 0$ and $x^{10} + y^{10} - 2z^{10} = 0$. What are their multiplicities? Explain why this is or isn't compatible with Bézout's theorem.*

Problem 9 (5 points) *Find the points in the complex* $\mathbb{P}^2_{\mathbb{C}}$ *on the curves* $z^2 - xy = 0$ *and* $x^{10} + y^{10} - 2z^{10} = 0$*. What are their multiplicities? Explain why this is or isn't compatible with Bézout's theorem.* HINT: Use the parameterization of $z^2 - xy = 0$ given by $[a, b] \to [a^2, b^2, ab]$.

Problems due October 2

Problem 10 (3 points) *Compute* $d\omega$ *where* $\omega = x^2dx + y^4dy$*. Let* $D = \{(x, y) \in \mathbb{R}^2 \big| x^2 + y^2 \leq 2\}$*. Compute* $\displaystyle\int_{\partial D} x^2dx + y^4dy$*.* HINT: Use Green's formula.

Problem 11 (2 points) *Compute* $d\omega$ *where* $\omega = ydx + xdy$*. Let* $D = \{(x, y) \in \mathbb{R}^2 \big| x^2 + y^2 \leq 2\}$*. Compute* $\displaystyle\int_{\partial D} ydx + xdy$*.*

Problem 12 (2 points) *Let* $D = \{z \in \mathbb{C} \big| |z| \leq 1\}$*. Compute* $\displaystyle\int_{\partial D} \frac{dz}{4z^2 - 1}$.

HINT: Note that $\displaystyle\frac{1}{4z^2 - 1} = \frac{1}{4}\left(\frac{1}{z - \frac{1}{2}} - \frac{1}{z + \frac{1}{2}}\right)$.

Problem 13 (3 points) *Find the Taylor series around* 0 *of* $\sqrt{1 - z}$*. What is the radius of convergence?*

Problems due October 16: In problems **??**, **??**, **??** we cover $\mathbb{P}^1$ with the two usual coordinate patches, $z \to [z, 1]$ and $w \to [1, w]$ with $w = 1/z$ on the overlap.

Problem 14 (2 points) *If* $p(z)$ *is the polynomial* $z^6 + z^3 + 1$ *what is the form of the rational 'function' it gives rise to expressed in terms of* w*, i.e.,* $p(1/w)$*? Regarded as a 'function' on* $\mathbb{P}^1$ *how many zeroes does* $p(z)$ *have and how many poles does it have?*

Problem 15 (3 points) *If* $r(z)$ *is the rational 'function'* $\displaystyle\frac{z^6 + z^3 + 1}{z^7 + 8z}$ *what is the form of the rational 'function' it gives rise to expressed in terms of* w*, i.e.,* $r(1/w)$*? Regarded as a 'function' on* $\mathbb{P}^1$ *how many zeroes does* $r(z)$ *have and how many poles does it have?*

Problem 16 (3 points) *If* $p(z)\mathrm{d}z$ *is a one-form with* $p(z) = z^2 + 1$ *what is the form of the rational one-form it gives rise to expressed in terms of* w*? Regarded as a one-form on* $\mathbb{P}^1$ *how many zeroes does* $p(z)\mathrm{d}z$ *have and how many poles does it have?*

Problem 17 (2 points) *Given the map* $A : \mathbb{C} \to \mathbb{C}$ *given by* $w = A(z) = z^3$*, what is the pullback* A^*dw*? What is the pullback* $A^*(\frac{\mathrm{d}w}{w})$*.*

Problems due October 30:

Problem 18 (4 points) *What is the partial fraction decomposition over the real numbers $\mathbb{R}$ of*

1. $\dfrac{x+3}{x^2+1}$?
2. $\dfrac{x^2+3}{x^2+1}$?
3. $\dfrac{x^3+x^2+3}{x^2+1}$?
4. $\dfrac{x^2+3}{(x^2+1)^2}$?

Problem 19 (4 points) *What is the partial fraction decomposition over the complex numbers $\mathbb{C}$ of*

1. $\dfrac{x+3}{x^2+1}$?
2. $\dfrac{x^3+x^2+3}{x^2+1}$?
3. $\dfrac{x^3+x^2+3}{(x^2+1)^2}$?
4. $\dfrac{x^2+3}{(x-1)^2(x+3)}$?

Problem 20 (3 points) *What is the divisor of*

1. $\dfrac{x+3}{x^2+1}$?
2. $(x-1)^4(x-2)^3(x-7)^8$?
3. $\dfrac{(x-1)^4(x-7)^4}{(x+1)^4(x+7)^8}$?

Problems due November 6: I strongly recommend you use a symbolic processing program for these problems. Set up the appropriate matrix for each problem. Compute the determinant and interpret the answer.

Problem 21 (3 points) *Compute the discriminant of x^3+ax+b.*

Problem 22 (3 points) *Compute the resultant of x^2-ax+b and x^2-Ax+B.*

Problem 23 (4 points) *Compute the resultant of $x^3 - ax + b$ and $x^2 - Ax + B$.*

Problem 24 (2 points) *What is the Euler characteristic of a figure eight?*

Problems due November 13 The next two problems give a standard example of an integral domain that is not a UFD.

Problem 25 (3 points) *In the domain $\mathbb{Z}[\sqrt{-3}]$, the norm of an element $a + b\sqrt{-3}$ is given by*

$$N(a + b\sqrt{-3}) = (a + b\sqrt{-3}) \cdot (a - b\sqrt{-3}) = a^2 + 3b^2.$$

1. *Show that $N((a + b\sqrt{-3}) \cdot (c + d\sqrt{-3})) = N(a + b\sqrt{-3}) \cdot N(c + d\sqrt{-3})$.*
2. *Show that $N(a + b\sqrt{-3}) = 1$ if and only if $a = \pm 1$ and $b = 0$.*
3. *Show that $N(a + b\sqrt{-3}) = 2$ is impossible.*

Problem 26 (3 points) *In this problem you can use the results of Problem* **??**.

1. *Show that any element $x + y\sqrt{-3} \in \mathbb{Z}[\sqrt{-3}]$ with $N(x + y\sqrt{-3}) = 4$ is irreducible, i.e., show there are not two elements of $a+b\sqrt{-3}, c+d\sqrt{-3} \in \mathbb{Z}[\sqrt{-3}]$, which are both not units with $x+y\sqrt{-3} = (a+b\sqrt{-3})\cdot(c+d\sqrt{-3})$.* `(Hint: If` $x + y\sqrt{-3} = (a + b\sqrt{-3}) \cdot (c + d\sqrt{-3})$`, then` $4 = N(x + y\sqrt{-3}) = N(a+b\sqrt{-3})\cdot N(c+d\sqrt{-3})$`, and thus either` $N(a+b\sqrt{-3}) = 1, N(c+d\sqrt{-3}) = 4$`; or` $N(a+b\sqrt{-3}) = 2, N(c+d\sqrt{-3}) = 2$`; or` $N(a+b\sqrt{-3}) = 4, N(c + d\sqrt{-3}) = 1$`. By the above either` $a + b\sqrt{-3}$ `or` $c + d\sqrt{-3}$ `is a unit.)`
2. *Show that $4 = 2\cdot 2 = (1+\sqrt{-3})(1-\sqrt{-3})$. Why does this show that 4 can not be uniquely factored, and in particular that $\mathbb{Z}[\sqrt{-3}]$ is not a UFD.*

I strongly recommend you use a symbolic processing program for the next problem. Set up the appropriate matrix and compute its determinant.

Problem 27 (4 points) *Using the resultant, eliminate z between the two cubics $x^3 + y^3 + z^3 = 0$ and $x^2y + xyz + yz^2 + z^3 = 0$. If you could factor resulting polynomial into factors that are linear in x and y, how could you use this to solve find the nine common zeroes of the above two cubics?*

4 First Test, October 2, 1996

Problems

In the following you must show your work. All $\mathbb{P}^1$'s and $\mathbb{P}^2$'s are over the complex numbers unless explicitly stated otherwise.

Problem 1 (12 points total) *We have the map $\phi : \mathbb{C}^2 \to \mathbb{P}^2$ given by $\phi(x, y) = [x, y, 1]$.*

1. *Which points of $\mathbb{P}^2$ are not in the image under ϕ of $\mathbb{C}^2$?*
2. *Is the map ϕ one-to-one on $\mathbb{C}^2$?*

Problem 2 (8 points) *What homogeneous polynomial $p(x, y, z)$ has the following partial derivatives:*

$$\frac{\partial p}{\partial x} = 4x^3 + 3zx^2, \quad \frac{\partial p}{\partial y} = 4y^3, \quad \frac{\partial p}{\partial z} = x^3.$$

Problem 3 (9 points) *Let $f(x, y) = x^3 + y^2 + x = 0$ define a curve in $\mathbb{C}^2$. Considering $\mathbb{C}^2$ as a subset of $\mathbb{P}^2$ by the map $(x, y) \to [x, y, 1]$, what is the homogeneous polynomial of the same degree as f which gives the equation f on $\mathbb{C}^2$?*

Problem 4 (10 points total) *We have the curve C defined by $x^3 + y^2 z = 0$.*

1. *What are the singular points of C?*
2. *How many distinct singular points are there for this equation?*

Problem 5 (12 points total) *State Bézout's theorem. Is there any conflict with the fact that the curves in the* **real** *projective plane $\mathbb{P}^2_{\mathbb{R}}$ defined by $x^2 + y^2 - z^2 = 0$ and $x^2 + y^2 - 4z^2 = 0$ have no points in common?*

Problem 6 (15 points total) *Let C be the curve in $\mathbb{P}^2$ defined by $p(x, y, z) = yz^3 - x^4 + 2x^2z^2 - z^4 = 0$.*

1. *What is the homogeneous equation of the tangent line L of the curve C in $\mathbb{P}^2$ at $[1, 0, 1]$? (SOME HELP: $\dfrac{\partial p(1,0,1)}{\partial x} = 0, \dfrac{\partial p(1,0,1)}{\partial y} = 1, \dfrac{\partial p(1,0,1)}{\partial z} = 0$.)*

2. *What points does this line L meet C in? What are their multiplicities?*

3. *Why is or is not this count of the number of points of intersection of L and C compatible with Bézout's theorem?*

Problem 7 (12 points) *Find the automorphisms* $\phi(z) = \dfrac{az+b}{cz+d}$ *of* $\mathbb{P}^1$ *that take* ∞ *to* ∞. *Which of these also take* 0 *to* 0*?*

Problem 8 (10 points total) *What is the power series expansion of* $\dfrac{1}{z}$ *around* $z = 1$*? What is the radius of convergence of this power series?*

Problem 9 (12 points total) *Compute the following line integrals over the unit circle,* $|z| = 1$. *(A convenient parameterization of the unit circle is given by* $z(t) = e^{it}$ *for* $0 \leq t \leq 2\pi$.*)*

1. $\displaystyle\int_{|z|=1} z \mathrm{d}z.$

2. $\displaystyle\int_{|z|=1} \overline{z} \mathrm{d}z.$

5 Second Test, November 15, 1996

Problem 1 (6 points) *If $r(z)$ is the rational 'function' $\dfrac{z^3}{z^2+8}$ what is the form of the rational 'function' it gives rise to expressed in terms of w, i.e., $r(1/w)$? Regarded as a 'function' on $\mathbb{P}^1$ how many zeroes does $r(z)$ have and how many poles does it have?*

Problem 2 (6 points) *If $p(z)\mathrm{d}z$ is a one-form with $p(z) = z^4$ what is the form of the rational one-form it gives rise to expressed in terms of w? Regarded as a one-form on $\mathbb{P}^1$ how many zeroes does $p(z)\mathrm{d}z$ have and how many poles does it have?*

Problem 3 (12 points) *What is the partial fraction decomposition over the real numbers $\mathbb{R}$ of*

1. $\dfrac{5x+3}{(x^2+5)^2}$?
2. $\dfrac{5x+3}{x^2+1}$?
3. $\dfrac{x^3+x+3}{x^2+1}$?

Problem 4 (16 points) *What is the partial fraction decomposition over the complex numbers $\mathbb{C}$ of*

1. $\dfrac{x^2}{x-1}$?
2. $\dfrac{x}{x^2+1}$?
3. $\dfrac{x^3-x+3}{x^2-1}$?
4. $\dfrac{x+1}{(x-1)^2}$?

Problem 5 (12 points) *What is the divisor on $\mathbb{P}^1$ of*

1. $\dfrac{x}{x^2-1}$?

2. $(x-2)^3(x-4)^2(x-7)$?

3. $\dfrac{(x-1)^4(x-3)^4}{(x+2)^2(x+9)^4}$?

Problem 6 (10 points) *You are given two real polynomials* $p(x) = x^2 - 1$ *and* $q(x) = (x-1)^3$.

1. *Find the greatest common divisor of* $p(x)$ *and* $q(x)$.
2. *Find the resultant of these two polynomials—an explicit numerical answer and a justification for it are required.*

Problem 7 (6 points) *Set up the matrix (i.e., you do not have to compute the determinant) to compute the discriminant of* $x^3 + 2x^2 + 3x + 4$.

Problem 8 (6 points) *Compute the resultant of* $x^2 + 1$ *and* $x^2 + 2$. *(For this problem you must compute a determinant. You must show your work, i.e., no credit will be given for simply writing down the determinant).*

Problem 9 (6 points) *You have a polynomial* $p(x, y) \in \mathbb{C}[x, y]$. *You know that there is no point* $(x, y) \in \mathbb{C}^2$ *with* $p(x, y) = 0$. *Moreover you know* $p(0, 0) = 1$. *Write down all such* $p(x, y)$ *explicitly. Justify your answer. Show the importance in the above of considering all complex solutions by giving an example of a nontrivial polynomial that has no zeroes on* $\mathbb{R}^2$.

Problem 10 (10 points) *The resultant needed to eliminate* y *between the* $1 + x^3 + y^2 + y^3 = 0$ *and* $x^2 + xy + y^2 + y^3 = 0$ *is* $p(x) = 1 + x - 3\,x^2 + 2\,x^3 + 5\,x^4 - 5\,x^5 + x^6 + 4\,x^7 - 3\,x^8 + x^9$. *Assume that you know how to find the complex solutions to any one variable polynomial with complex coefficients, e.g.,* $p(x) = 0$. *Explain (with a justification) how you might find the common zeroes in* $\mathbb{C}^2$ *of* $1 + x^3 + y^2 + y^3 = 0$ *and* $x^2 + xy + y^2 + y^3 = 0$. *(Do not worry about multiplicities of solutions.)*

Problem 11 (10 points) *What is the Euler characteristic of the Nephroid of Freeth?*

`A picture of the nephroid of freeth was included here`