1 First Test

Mathematics 423: Test I: February 25, 1998

Name:

To receive credit you must show your work.

Problem Number	Maximum Points	Points attained
1	8	
2	8	
3	16	
4	6	
5	6	
6	6	
7	6	
8	10	
9	10	
10	10	
11	6	
12	8	
TOTAL	100	

Some Useful Results

The following results may be of some use. You can assume them in any argument you need to give.

Theorem 1.1 (Neville's Recursion Formula) *Let $x_0, \ldots, x_n$ be distinct real numbers in the interval, $[a, b]$. Let f be a real valued function on $[a, b]$, and let $P(x)$ denote the unique polynomial of degree $\leq n$ such that $f(x_i) = P(x_i)$ for $i = 0, \ldots, n$. For $c = 0, \ldots, n$, let $P_{\hat{c}}(x)$ denote the unique polynomial of degree $\leq n-1$ such that $P_{\hat{c}}(x_i) = f(x_i)$ for $0 \leq i \leq n$ with $i \neq c$. Then for two distinct points x_j and x_k in the set $\{x_0, \ldots, x_n\}$:*

$$P(x) = \frac{(x - x_j)P_{\hat{j}}(x) - (x - x_k)P_{\hat{k}}(x)}{x_k - x_j}$$

or equivalently:

$$P(x) = \frac{x - x_j}{x_k - x_j} P_{\hat{j}}(x) + \frac{x - x_k}{x_j - x_k} P_{hatk}(x).$$

Theorem 1.2 (Hermite Interpolation Error Formula) *Let $x_0, \ldots, x_n$ be distinct real numbers in the interval, $[a, b]$. Let f be an $2n+2$ times continuously differentiable function on $[a, b]$. Let $H(x)$ denote the the unique polynomial of degree $\leq 2n+1$ such that for $0 \leq i \leq n$, $f(x_i) = H(x_i)$ and $f'(x_i) = H'(x_i)$. Then for each $x \in [a, b]$, there exists a point $\xi \in (a, b)$ such that*

$$f(x) - H(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}(x - x_0)^2 \cdots (x - x_n)^2.$$

Theorem 1.3 (Euler Summation Formula) *Let f be an $2m+2$ times continuously differentiable function on $[a, b]$. Let $h := \dfrac{b-a}{n}$ for some integer $n \geq 1$. Let $x_j := a + jh$ for $j = 0, \ldots, n$. Let B_j denote the j-th Bernoulli number. There exists $\xi \in (a, b)$ such that*

$$\begin{aligned}\left(f(a) + \sum_{j=1}^{j=n-1} 2f(x_j) + f(b)\right)\frac{h}{2} &= \int_a^b f(x)dx + \sum_{j=1}^{m}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]B_{2j}\frac{h^{2j}}{(2j)!} \\ &+ f^{2m+2}(\xi)B_{2m+2}\frac{h^{2m+2}(b-a)}{(2m+2)!}.\end{aligned}$$

Problems

In the following you must show your work.

Problem 1 (8 points total) *Perform the following computations in a) 3 digit rounding arithmetic; and compute b) the absolute error, and c) the relative error in each case:*

1. $(121 - 0.3) - 119$;
2. $(121 - 119) - 0.3$.

Put your answers in the table below.

expression to be evaluated	3 digit rounding answer *(2 points each)*	absolute error *(1 point each)*	relative error *(1 point each)*
$(121 - 0.3) - 119$			
$(121 - 119) - 0.3$			

Problem 2 (8 points total) *Perform the following computations in a) 3 digit chopping arithmetic; and compute b) the absolute error, and c) the relative error in each case:*

1. $(102 - 0.3) - 0.7$;
2. $102 - (0.3 + 0.7)$.

Put your answers in the table below.

expression to be evaluated	3 digit rounding answer **(2 points each)**	absolute error **(1 point each)**	relative error **(1 point each)**
$(102 - 0.3) - 0.7$			
$102 - (0.3 + 0.7)$			

Problem 3 (16 points total) *A person wishes to find a zero of* $f(x) = e^x - 3x$ *for* $0 \leq x \leq 1$. *That person decides to use the bisection method to accomplish this.* Assume for simplicity that you are using exact arithmetic, i.e., that the errors of computer arithmetic play no role here.

5 pts: *State a theorem and show that it applies to guarantee that* $f(x)$ *has a zero on* $[0, 1]$.

5 pts: *The first approximation to a solution by the bisection method is* 0.5 *with* $f(0.5) = e^{0.5} - 3 \cdot 0.5 \approx 0.15$. *What is the second approximation to a solution by the bisection method?*

6 pts: *You would like to find an approximation to a zero of* $f(x)$ *on* $[0, 1]$ *with an absolute error of no more than* 0.001. *Using the bisection method as outlined in the previous part of this problem, and the error estimate for the bisection method, which is the smallest integer* n *for which you know that on the* n*-th approximation you will be within* 0.001 *of the correct answer. An answer without justification by the error estimate for the bisection method will receive no credit.*

Problem 4 (6 points total) *A person wishes to find a zero of* $f(x) = e^x - 3x$ *for* $0 \leq x \leq 1$*. That person decides to use the secant method for finding a solution of the equation* $f(x) = 0$ *on the interval,* $[0, 1]$*.* Assume for simplicity that you are using exact arithmetic, i.e., that the errors of computer arithmetic play no role here. *What is the first approximation to a solution in the open interval* $(0, 1)$ *by the secant method?*

Problem 5 (6 points total) *A person wishes to find a zero of* $f(x) = e^x - 3x$ *for* $0 \leq x \leq 1$*. That person decides to use Newton's method (also known as the Newton-Raphson method) for finding a solution of the equation* $f(x) = 0$ *on the interval,* $[0, 1]$*.* Assume for simplicity that you are using exact arithmetic, i.e., that the errors of computer arithmetic play no role here. *Write down iteration formula that Newton's method gives for solving* $f(x) = 0$*, and using* 1.0 *as a starting guess find the first approximation to a solution of* $f(x) = 0$ *given by this formula.*

Problem 6 (6 points total) *Does* $p_n = 10^{-3^n}$ *with* $n = 1, 2, 3, \ldots$ *converge to zero of order* 3*? To receive credit you must justify your answer, i.e., show it converges of order* 3 *to zero or show why it does not converge of order* 3 *to zero.*

Problem 7 (6 points total) *Write down the Lagrange form* (not the Newton form) *of the interpolating polynomial of degree* ≤ 2 *with the value* 0 *at* $x_0 = 0$*, the value* 2 *at* $x_1 = 1$*, and the value* 3 *at* $x_2 = 3$*.*

Problem 8 (10 points total) *Given the function* $f(x) = x^3$*:*

7 pts: *Compute the divided differences,* $f[0], f[0, 2], f[0, 2, 3]$;

3 pts: *Write down the interpolating polynomial,* $p_2(x)$*, of degree* ≤ 2 *for* $f(x)$ *with the node points* $x_0 = 0$*,* $x_1 = 2$*,* $x_2 = 3$ *using Newton's interpolatory divided-difference formula, i.e., by using the Newton polynomial built out of divided differences* (not the Lagrange form).

Problem 9 (10 points total) *Neville's method is used to approximate* $f(1)$ *giving the following table:*

$$\begin{array}{llll} x_0 = 0 & P_0 = 0 & & \\ x_1 = 2 & P_1 = 16 & P_{01} = 8 & \\ x_2 = 3 & P_2 & P_{12} & P_{123} = 8 \end{array}$$

Determine $P_2 = f(3)$*.*

Problem 10 (10 points) *Let* $f(x) = \sin\left(\frac{\pi}{2}x\right)$. *Let* $H_3(x)$ *denote the unique polynomial of degree* ≤ 3 *such that* $f(0) = H_3(0)$, $f'(0) = H_3'(0)$, $f(1) = H_3(1)$, $f'(1) = H_3'(1)$.

5 points: *Compute the coefficients* a_0, a_1, a_2, a_3, *of* H_3 *in the Newton divided difference form* $H_3(x) := a_0 + a_1x + a_2x^2 + a_3x^2(x-1)$.

5 points: *Use the error term in theorem (**??**) to estimate* $|H_3(0.5) - f(0.5)|$. You may use the upper bound $\max\limits_{[0,1]} |f^{(4)}(x)| \leq 6.4$.

Problem 11 (6 points) *Let* $f(x)$ *be a* $2m+2$ *times differentiable function defined on the real line. Assume that* $f(x)$ *is* periodic *with period* 4, *i.e.,* $f(x+4) = f(x)$ *for all real points* x. *Prove that the trapezoidal approximation on* $[0,4]$ *is an order* $2m+2$ *method, i.e., show that with* $h = 4/n$, *and* $x_j = jh$ *for* $j = 0, \ldots, n$:

$$\left(f(0) + \sum_{j=1}^{j=n-1} 2f(x_j) + f(4)\right)\frac{h}{2} = \int_0^4 f(x)dx + O(h^{2m+2}).$$

Problem 12 (8 points total) *Consider a four times continuously differentiable real valued function* $f(x)$ *on* $[0,1]$. *Assume that the trapezoidal approximations for* $n = 1, 2$ *are* $2, 2.3$ *respectively. What is the Richardson extrapolation estimate to* $\int_0^1 f(x)dx$ *that the Romberg method of integration makes using these trapezoidal approximations?*

2 Second Test

Mathematics 423: Test II: April 8, 1998

Name: ....

To receive credit you must show your work.

Problem Number	Maximum Points	Points Lost
1	10	
2	10	
3	10	
4	15	
5	11	
6	12	
7	10	
8	8	
9	14	
TOTAL LOST		

Runge-Kutta of Order Four For the ordinary differential equation $y' = f(t, y)$ on $[a, b]$ with initial condition $y(a) = \alpha$ and stepsize h we have

$$
\begin{aligned}
w_0 &= \alpha \\
k_1 &= hf\left(t_i, w_i\right) \\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right) \\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right) \\
k_4 &= hf\left(t_i + h, w_i + k_3\right) \\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

Problems

In the following you must show your work.

Problem 1 (10 points)

1. *Use Euler's method to approximate the solution of the following initial value problem:*

$$y' = ty;\ \ 1 \le t \le 2;\ \ y(1) = \sqrt{e},\ \textit{with}\ h = 1.0.$$

2. *The actual solution of the above differential equation is* $y(t) = e^{\frac{t^2}{2}}$. *Compute the absolute error of the approximate solution at* $t = 2.0$ *found in part 1 of this problem.*

Problem 2 (10 points)

1. *Use Taylor's method of order two to approximate the solution of the initial value problem as in Problem* **??***:*

$$y' = ty;\ 1 \leq t \leq 2.0;\ y(1) = \sqrt{e}, \textit{with } h = 1.0.$$

2. *The actual solution of the above differential equation is* $y(t) = e^{\frac{t^2}{2}}$. *Compute the absolute error of the approximate solution at* $t = 2.0$ *found in part 1 of this problem.*

Problem 3 (10 points)

1. **5 points:** *Use the Runge-Kutta method of order four to approximate the solution of the initial value problem as in Problem* **??***:*

$$y' = ty;\ 1 \leq t \leq 2.0;\ y(1) = \sqrt{e}, \textit{with } h = 1.0.$$

(See page one for some useful formulae!)

2. **5 points:** *The actual solution of the above differential equation is* $y(t) = e^{\frac{t^2}{2}}$. *Compute the absolute error of the approximate solution at* $t = 2.0$ *found in part 1 of this problem.*

Problem 4 (15 points) *Consider the difference equation:*

$$y_n = 7y_{n-1} - 10y_{n-2} + 4.$$

1. **10 points:** *Solve the above difference equation, i.e., find* y_n *for* $n \geq 0$, *subject to the initial conditions* $y_0 = 1$, $y_1 = 4$.
2. **5 points:** *Find* $\lim_{n\to\infty} \frac{y_{n+1}}{y_n}$.

Problem 5 (12 points) *Let* $v = (2, 2, 4)$ *be the vector in* $\mathbb{R}^3$. *Compute* $||v||_1$, $||v||_2$, *and* $||v||_\infty$.

Problem 6 (11 points) *Consider the solution $y(t)$ of the initial value problem:*

$$y'' - xy' + 4y = 0; \quad -1 \le x \le 1$$

with initial values $y(-1) = -1, y'(-1) = 2$. Rewrite the ordinary differential equation as a 1st order system of ordinary differential equations.

Problem 7 (8 points) *Let $< f(x), g(x) >= \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}dx$ be an inner product on V_5, the vector space of polynomials of degree ≤ 5 on $[-1,1]$. It is known that the Chebychev polynomials*

$$T_0(x), T_1(x), T_2(x), T_3(x), T_4(x), T_5(x)$$

form an orthogonal basis of V_5 relative to this inner product. Moreover it is known that $< T_n(x), T_n(x) >= \frac{\pi}{2}$ for $n > 0$ and $< T_0(x), T_0(x) >= \pi$.

Assume that $f(x) = T_1(x)+4T_2(x)-4T_3(x)+T_4(x)$. Find $< f(x), f(x) >$.

Problem 8 (10 points) *Let $||g(x)|| = \int_0^1 g(x)^2 dx$ be the norm associated to the inner product $< f(x), g(x) >= \int_0^1 f(x)g(x)dx$ on the polynomials on $[0,1]$.*

1. **7 points:** *Find the linear least squares polynomial approximation to $f(x) = x^2 + 2$ on $[0,1]$, i.e., find the polynomial $h(x)$ of degree ≤ 1 such that $||h(x) - f(x)||$ is minimum.*
2. **3 points:** *What is $\int_0^1 (h(x) - f(x))(3x+7)dx$? If you do this without a computation explain why you know the number you wrote down is the answer.*

Problem 9 (14 points) *Consider the problem of finding a solution of $f(x) = 0$ where f is a system of polynomials in $x = (x_1, x_2) \in \mathbb{R}^2$*

$$f(x) = \begin{pmatrix} 2x_1 + x_2 + x_2^2 - x_1x_2 - 1 \\ x_1^2 - 2x_1 + 2x_2 \end{pmatrix}$$

1. **7 points:** *Compute the Jacobian of f.*
2. **7 points:** *Use Newton's method with $x^0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to compute the first approximation x^1 to a solution of $f(x) = 0$.*

3 Final

Mathematics 423: Final Exam: May 7, 1998

Name: ______________________________

Problem Number	Maximum Points	Points Lost
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
10	10	
11	10	
12	10	
13	12	
TOTAL		

Neville's Recursion Formula: Let $x_0, \ldots, x_n$ be distinct real numbers in the interval, $[a, b]$. Let f be a real valued function on $[a, b]$, and let $P(x)$ denote the unique polynomial of degree $\leq n$ such that $f(x_i) = P(x_i)$ for $i = 0, \ldots, n$. For each $c = 0, \ldots, n$, let $P_{\widehat{c}}(x)$ denote the unique polynomial of degree $\leq n-1$ such that $P_{\widehat{c}}(x_i) = f(x_i)$ for $0 \leq i \leq n$ with $i \neq c$. Then for two distinct points x_j and x_k in the set $\{x_0, \ldots, x_n\}$:

$$P(x) = \frac{(x - x_j)P_{\widehat{j}}(x) - (x - x_k)P_{\widehat{k}}(x)}{x_k - x_j}$$

or equivalently:

$$P(x) = \frac{x - x_j}{x_k - x_j} P_{\widehat{j}}(x) + \frac{x - x_k}{x_j - x_k} P_{hatk}(x).$$

Problems

To receive credit you must show your work.

Problem 1 (12 points total) *Perform the following computations in a) 3 digit rounding arithmetic; and compute b) the absolute error, and c) the relative error in each case:*

1. $(100 - 0.6) + 0.6$;
2. $(99.8 + 0.6) - 0.6$.

Put your answers in the table below.

expression to be evaluated	3 digit rounding answer *(2 points each)*	absolute error *(2 points each)*	relative error *(2 points each)*
$(100 - 0.6) + 0.6$			
$(99.8 + 0.6) - 0.6$			

Problem 2 (12 points total) *A person decides to use Newton's method (also known as the Newton-Raphson method) to find a zero of $f(x) = x^2 - 2$ for $1 \leq x \leq 2$.* Assume that you are using exact arithmetic, i.e., that the errors of computer arithmetic play no role here. *Write down iteration formula that Newton's method gives for solving $f(x) = 0$, and using 2.0 as a starting guess find the first and the second approximation to a solution of $f(x) = 0$ given by this formula.*

Problem 3 (12 points total) *Given a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$, $f(1) = 0$, $f(2) = 4$, $f(3) = 0$, write down in Lagrange form* (not the Newton form) *of the interpolating polynomial of degree ≤ 3 with values agreeing with f at $x = 0, 1, 2, 3$.*

Problem 4 (12 points total) *Given a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$, $f(1) = 0$, $f'(1) = 2$, $f(2) = 4$:*

8 pts *Write down the divided differences, $f[0], f[0,1], f[0,1,1], f[0,1,1,2]$;*

4 pts *Write down the Hermite interpolating polynomial of degree ≤ 3 for f at the values $0, 1, 1, 2$ using Newton's interpolatory divided-difference formula, i.e., by using the Newton polynomial built out of divided differences.*

Problem 5 (12 points total) *Let $S(x)$ be a free cubic spline defined on $[0, 2]$ with nodes $x_0 = 0, x_1 = 1, x_2 = 2$, and $S(0) = 0$, $S(1) = 0$ and $S(2) = 4$. Assume that $S(x)$ is given by the cubic polynomials $S_0(x) = a - x + x^3$ on $[0, 1]$, and by $S_1(x) = b(x-1) + c(x-1)^2 - (x-1)^3$ on $[1, 2]$. Find the real numbers a, b, c, i.e., find the actual numbers.* [Recall the free splines satisfy the conditions that $S_0''(0) = 0 = S_1''(2)$.]

Problem 6 (12 points total) *Neville's method is used to approximate $f(2)$ giving the following table:*

$$\begin{array}{llll} x_0 = 0 & P_0 = f(0) = 1 & & \\ x_1 = 1 & P_1 = f(1) = 2 & P_{01} = 3 & \\ x_2 = 3 & P_2 = f(3) = ? & P_{12} = 15 & P_{012} = 11 \end{array}$$

Determine $P_2 = f(3)$.

Problem 7 (12 points total) *You would like to integrate $\int_1^2 [\ln(x)]^2 \mathrm{d}x$. The trapezoidal approximation for $n = 1$ is 0.240, and for $n = 2$ it is 0.202. What is the Richardson extrapolation estimate to $\int_1^2 [\ln(x)]^2 \mathrm{d}x$ that the Romberg method of integration makes using these trapezoidal approximations?*

Problem 8 (12 points)

1. *Set up the Taylor method of order 2 to solve $y' = y^2 t^2$ for $t \in [0, 1]$ with $h = 0.25$, $y(0) = 1$.*

2. *Set up the Euler method to solve the same initial value problem and with $h = 0.25$.*

Problem 9 (12 points) *Consider the initial value problem:*

$$y''' + 2y'' + 4y' + 8y = t^4; \quad 0 \leq t \leq 3$$

with initial values $y(0) = 1, y'(0) = 2$, $y''(0) = 0$. Rewrite the ordinary differential equation as a 1st order system of ordinary differential equations. [Do not forget to write down the initial conditions for the system.]

Problem 10 (10 points) *Give an argument based on the Gershgorin Theorem to show that the matrix* $\begin{bmatrix} 5 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}$ *is invertible.* Hint: You can use the fact that a matrix A is invertible if and only if 0 is not an eigenvalue of A.

Problem 11 (10 points) *What is the infinity norm* $||A||_\infty$ *of the matrix*

$$A = \begin{bmatrix} -10 & 1 & 2 \\ 1 & 0 & 2 \\ 4 & 2 & 2 \end{bmatrix}$$

Problem 12 (10 points) *What is the condition number (using the infinity norm) of the matrix*

$$\begin{bmatrix} 4 & 1 \\ 0 & 0.25 \end{bmatrix}$$

Problem 13 (12 points) *Consider the boundary value problem:*

$$y'' = y + x; \quad 0 \le x \le 2$$

with boundary values $y(0) = 1, y(2) = 2$. *Solve this equation using finite differences with the node points* $x_0 = 0, x_1 = 1, x_2 = 2$.