Math 425 **Spring 1995**

Review for Test No. 1

1. Complex numbers. Conjugates. Arithmetical operations on complex numbers. Absolute value and the triangle inequality. Infinity. Polar form. De Moivre's formula. n-th roots of complex numbers.

Examples:

1.1 Calculate the following expressions:

(a) $\dfrac{1 + i \tan \alpha}{1 - i \tan \alpha}$, α - a real number, (b) $\dfrac{(1 + i)^9}{(1 + i)^7}$.

1.2 Find all complex numbers z, which are conjugate to their cubes: $z = \overline{z^3}$.

1.3 Calculate $\left(\dfrac{1-i}{1+i}\right)^{1995}$.

1.4 Find all third roots of $-2 + 2i$

1.5 Use De Moivre's formula to prove that $\tan 6\theta = \dfrac{6 \tan \theta - 20 \tan^3\theta + 6 \tan^5\theta}{1 - 15 \tan \theta + 15 \tan^5\theta - \tan^6\theta}$

2. Geometry of complex numbers. Lines, circles, the Appolonius circles.

2.1 Find the loci of the equations or inequalities and draw then on a diagram:

(a) $|z - 2| + |z + 2| = 5$, (b) $|z - 2| - |z + 2| > 3$, (c) $|z| = \mathrm{Re}(z) + 1$

(d) $0 < \mathrm{Re}(iz) < 2$

3. Topology of complex numbers. Open and closed sets. Connected sets.
Simply connected sets. Limit of a sequence, an limit of a function. Continuity.
Infinite series and convergence tests. Power series. Radius of convergence. Hadamard's formula $R = \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$. Using other tests to determine the radius of convergence.

Examples:

3.1 Which of the following sets are open or closed? Find their boundaries. Are they connected, simply connected? Draw a diagram

(a) $\{z \mid |z - 2| - |z + 2| = 3\}$, (b) $\{z \mid 0 < |z - 2| + |z + 2| \le 5\}$,

(c) $\{z \mid |z| < 2 |z + 1|$

3.2. Which of the following functions has a limit at 0?

(a) $\frac{\mathrm{Re}(z)}{|z|}$, (b) $\frac{z}{|z|}$, (c) $\frac{\mathrm{Re}(z^2)}{|z^2|}$,

(d) $\frac{z\mathrm{Re}(z)}{|z|}$

3.3 Find the radius of convergence of the series

(a) $\sum_{n=0}^{\infty} n^k z^n$, where k is a positive integer. (b) $\sum_{n=0}^{\infty} n^n z^n$,

(c)) $\sum_{n=0}^{\infty} \frac{2^n}{n!} z^n$,

4. The exponential, logarithm and trigonometric functions.

4.1. Find the real and imaginary parts of:

(a) $\cos(2 + i)$, (b) $\sin 2i$, c) $\sinh(e^{i})$.

5. Line integrals and Green's theorem.

5.1. Evaluate the integrals $I = \int_\gamma \mathrm{Re}(z)\, dz$ and $J = \int_\gamma \mathrm{Im}(z)\, dz$ along

(a) The semicircle $|z| = 1$, $0 \leq \arg(z) \leq \pi$, with initial point $z = -1$

(b) The line segment joining the points $z = 0$ to the point $z = 2 + i$.

5.2 Use Green's theorem to prove that $\int_\gamma z\, dz = 0$, where γ is the circle $|z|=1$.

Verify that the identity $\frac{\partial f}{\partial z} + i \frac{\partial f}{\partial y} = 0$ holds also for the function $f(z) = \frac{1}{z}$. Why then

cannot the prove be used in this case. Note that $\int_\gamma \frac{1}{z}\, dz = 2\pi i$