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Math 431 Exam III On Approximation of Irrational Numbers

by Pit-Mann Wong

Problem I. Find a simple continued fraction expansion of 1729/625.

Problem II. The simple continue fraction $p/q = [a_0, a_1, ..., a_n]$ of a rational number is unique if $a_n = 1$. If $a_n \neq 1$ show that $p/q = [a_0, a_1, ..., a_n - 1, 1]$.

By definition the k-th convergent of a simple continue fraction $p/q = [a_0, a_1, ..., a_n]$ is given inductively by

$$C_k = \frac{p_k}{q_k} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}$$

with $p/q = C_n$.

Proposition 1. *Let $[a_0, a_1, ..., a_n]$ be a simple continued fraction. Write the k-th convergent $C_k = p_k/q_k$ where p_k, q_k are relatively prime then $p_0 = a_0, q_0 = 1$ and for $k \geq 1$*

$$\begin{pmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{pmatrix} = \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{k-1} & q_{k-1} \\ p_{k-2} & q_{k-2} \end{pmatrix}.$$

Moreover, we have

$$\begin{cases} p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}, \\ p_k q_{k-2} - q_k p_{k-2} = (-1)^k a_k. \end{cases}$$

The preceding proposition may be used to solve the Diophantine equations:

$$Ax - By = \pm 1$$

provide that A and B are relatively prime.

Example 2. To solve the equation $257x - 97y = -1$, first find the ssimple continue fraction expansion of 257/97:

$$\frac{257}{97} = [2, 1, 1, 1, 5, 1, 4]$$

and its convergents $2, 3, 5/2, 8/3, 45/17, 53/20, 257/97$. Applying proposition 1 with $k = 6$ yields

$$257 \cdot 20 - 97 \cdot 53 = (-1)^{6-1} = -1$$

so $x = 20$ and $y = 53$ is a solution. If we ant to solve the equation $257x - 97y = 1$ then we have to replace the expansion by (see Problem II):

$$\frac{257}{97} = [2, 1, 1, 1, 5, 1, 3, 1]$$

with convergents $2, 3, 5/2, 8/3, 45/17, 53/20, 204/77, 257/97$. proposition 1 with $k = 7$ yields:

$$257 \cdot 77 - 97 \cdot 204 = (-1)^{7-1} = 1$$

thus $x = 77, y = 204$ is a solution.

Problem III. Solve the equations (1) 417 x - 172 y = 1, $(2) 1163x - 815y = 1$.

Example 3. Let $a/b = [a_0, a_1, ..., a_n, a_n, ..., a_1, a_0], a_i > 0$ for all i. A continue fraction of this form is said to be symmetric. Let $C_k = p_k/q_k$ be the k-th convergent how that $\{p_k\}, \{q_k\}$ are increasing sequences and

$$\begin{cases} p_n/p_{n-1} = [a_n, ..., a_1, a_0], \\ q_n/p_{n-1} = [a_n, ..., a_1]. \end{cases}$$

Proof. By Proposition 1, $p_k = a_k p_{k-1} + p_{k-2}$ which implies that p_k increases as each term on the right is positive. Moreover, we get

$$\frac{p_n}{p_{n-1}} = a_n + \frac{p_{n-2}}{p_{n-1}}$$

hence the first term of the continue fraction expansion for p_n/p_{n-1} is a_n with fractional part p_{n-2}/p_{n-1}. The next term of the expansion is obtained from $p_{n-1} = a_{n-1}p_{n-2} + p_{n-3}$, dividing through by p_{n-2} yields

$$\frac{p_{n-1}}{p_{n-2}} = a_{n-1} + \frac{p_{n-3}}{p_{n-2}}$$

which shows that the second term of the continue fraction expansion for p_n/p_{n-1} is a_{n-1} with fractional part p_{n-3}/p_{n-2}. It is clear that this process may be continued until we arrive at $p_0 = a_0$ with no fractional part. The proof is entirely analogous for q_k. QED

Example 4. If the convergents of a simple continue fraction are given by p_k/q_k then

$$(p_k^2 - q_k^2)(p_{k-1}^2 - q_{k-1}^2) = (p_k p_{k-1} - q_k q_{k-1})^2 - 1.$$

Proof. Squaring the right hamd side above yields

$$RHS = p_k^2 p_{k-1}^2 + q_k^2 q_{k-1}^2 - 2p_k p_{k-1} q_k q_{k-1} - 1.$$

On the other hand, the left hand side above equals

$$LHS = p_k^2 p_{k-1}^2 + q_k^2 q_{k-1}^2 - p_k^2 q_{k-1}^2 - q_k^2 p_{k-1}^2.$$

By the second part of Proposition 1,

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$$

and squaring yields

$$p_k^2 q_{k-1}^2 + q_k^2 p_{k-1}^2 - 2p_k p_{k-1} q_k q_{k-1} = 1$$

which implies that $RHS = LHS$. QED

Example 5. Let $a/b = [a_0, a_1, ..., a_n, a_n, ..., a_1, a_0] = p_{2n+1}/q_{2n+1}, a_i > 0$ for all i. Show that $p_{2n+1} = p_n^2 + p_{n-1}^2$ and $q_{2n+1} = p_n q_n + p_{n-1} q_{n-1}$.

Proof. Let $x = [a_n, ..., a_1, a_0]$ then

$$\frac{a}{b} = [a_0, a_1, ..., a_n, x] = (n+1) - \text{st convergent of } [a_0, a_1, ..., a_n, x] = \frac{xp_n + p_{n-1}}{xq_n + q_{n-1}}.$$

By example 3 $p_n/p_{n-1} = x$ hence the preceding can also be expressed as

$$\frac{p_{2n+1}}{q_{2n+1}} = \frac{a}{b} = \frac{p_n^2 + p_{n-1}^2}{p_n q_n + p_{n-1} q_{n-1}}.$$

QED

Example 6. Let $a/b = [a_0, a_1, ..., a_n, a_n, ..., a_1, a_0]$ be a symmetric continue fraction. Show that $p_{2n} = b$ and $a|b^2 + 1$.

Proof. Since there are $2n+2$ terms in the simple fraction $a/b = C_{2n+1} = p_{2n+1}q_{2n+1}$ hence $p_{2n+1} = a, q_{2n+1} = b$. By Example 3 and the condition that the continue fraction is symeetric we onclude that

$$\frac{p_{2n+1}}{p_{2n}} = [a_0, a_1, ..., a_n, a_n, ..., a_1, a_0] = \frac{a}{b}$$

hence $p_{2n} = b$. By the second part of Proposition 1,

$$p_{2n+1}q_{2n} - q_{2n+1}p_{2n} = (-1)^{2n+1-1} = 1$$

hence $aq_{2n} - b^2 = 1$. QED

Problem IV. Let $a/b = [a_0, a_1, ..., a_{n-1}, a_n, a_{n-1} ..., a_1, a_0]$ show that $a|b^2 - 1$.

Problem V. Recall that the Fibonacci numbers are defined by the conditions $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Show that

(a) The simple continue fraction expansion of $f_{n+1}/f_n = [1, ..., 1]$ (n ones).

(b) What identities can you derive for the Fibonaci sequences from Examples 5, 6 and problem IV?

(c) Conclude that any thre consecutive Fibonacci numbers have no common factor.

Definition 6. A rational number $p/q, p.q$ relatively prime is said to be a good (resp. best) approximation of an irrational number α if

$$|\alpha - \frac{p}{q}| < |\alpha - \frac{p^{'}}{q^{'}}| \text{ (resp. } |q\alpha - p| < |q^{'}\alpha - p^{'}|)$$

for all rational number $p^{'}/q^{'}$ with $p^{'}, q^{'}$ relatively prime and $q^{'} < q$.

Proposition 7. *If p/q is a best rational approximation of an irrational number α then it is also a good approximation.*

Theorem 8. *A raional number is a best approximation of an irrational number α if and only if it is a convergent of the simple continuous frction expansion of α.*

Example 9. Rational numbers with respective denominators $1, 2, 3, 4, 5, 6$ approximating $\pi \sim 3.1415927$ are respectively $3/1, 7/2, 10/3, 13/4, 16/5, 19/6$. The approximation 3 is a best approximation, $7/2, 10/3$ are not good approximations, $13/4, 16/5, 19/6$ are good approximations but not best. Thus the converse of Proposition 1 is not valid. The simple continuous fraction expansion of π is given by $[3, 7, 15, 1, 292, 1, ...]$. The convergents are given by

$$\begin{aligned}
C_0 &= 3, \\
C_1 &= 3 + \frac{1}{7} = \frac{22}{7}, \\
C_2 &= 3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106}, \\
C_3 &= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = \frac{355}{113}, \\
C_4 &= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = \frac{355}{113}, \\
C_5 &= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292}}}} = \frac{111613}{37204}.
\end{aligned}$$

Corollary 10. *If a rational number $p/q, p, q$ relatively prime satisfyies*

$$|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$$

then p/q is a convergent of α.

Corollary 11. *A number α is irrational if and only if the inequality*

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}$$

admits ininitely many rational solutions $p/q, p, q$ relatively prime.

Problem VI. In Example 9 verify that The approximation 3 is a best approximation, $7/2, 10/3$ are not good approximations, $13/4, 16/5, 19/6$ are good approximations but not best.