Math 436

Take-home problem for midterm

Due at 12:50 p.m., October 13, 1999

You may consult your course notes, homework, and the textbook. You may use Maple, Matlab or Mathematica as an experimental tool on any part of the problem. If you do other than on part g), please indicate how on your exam. You may not consult any other books or notes. You may not discuss the exam with anyone except me.

The in-class part of the exam will be on October 13. It will be closed book, but you may bring either a sheet of paper ($8\frac{1}{2}'' \times 11''$) with notes on one side or a $3'' \times 5''$ card with notes on both sides to the exam.

You must justify your answers. If you cannot do one part, you may assume the result of that part for the remaining parts.

Let

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x).$$

a) Show that for $0 < x \leq \pi$

$$K_N(x) = \frac{1}{2\pi(N+1)} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x}.$$

(**Hint:** Multiply the numerator and denominator of D_n by $2\sin\frac{x}{2}$ and convert products of sines to differences of cosines.)

b) Show that $K_N \geq 0$.

c) Show that $\int_{-\pi}^{\pi} K_N(x)\,dx = 1$.

d) Show that

$$K_n(x) \leq \frac{1}{\pi(N+1)} \cdot \frac{1}{1 - \cos\delta}$$

if $0 < \delta \leq |x| \leq \pi$.

e) Suppose f is piecewise smooth and 2π-periodic. Let $s_n(f;x)$ denote the nth partial sum of the Fourier series of f and let

$$\sigma_N = \frac{s_0 + s_1 + \ldots s_N}{N+1},$$

so σ_N is the arithmetic mean of the first $N+1$ partial sums. Show that

$$\sigma_N(f;x) = \int_{-\pi}^{\pi} f(x-t) K_N(t)\, dt.$$

f) Prove Fejér's theorem:

Theorem 1 *If f is continuous on $\mathbf{R}$ and periodic with period 2π then*

$$\sigma_N(f;x) \to f(x)$$

uniformly on $[-\pi, \pi]$.

(**Hints:** Your proof will require the results of b)-d). You will also need to use the fact that a continuous function g on a closed interval I is *uniformly continuous*: given $\epsilon > 0$ there is $\delta > 0$ such that $|g(u) - g(v)| < \epsilon$ if $u, v \in I$ and $|u - v| < \delta$.)

g) Graph K_N (on a computer) for $N = 5,\ 10,\ 50$. (Be sure to have the plot command compute enough points if you use Maple.) Compare with the graph of D_N.