Mathematics 438
Fall Semester 2000
Final Exam (take-home)
Due December 15, 2000

This Exam is due December 15, 2000, 10:00 am.

Please hand in your answers to Professor Cao on or before December 15, 2000, 10:00 am.

Sign the pledge: "On my honor, I have neither given nor received unauthorized aid on this Exam":

______________________________ **GOOD LUCK**

1. The equation $r = e^{-\theta}$ in polar coordinates describes a set of points in the plane as θ goes from 0 to ∞.

(a) Describe this set by a parametrized curve $\alpha : [0, \infty) \to \mathbb{R}^2$ with θ as the parameter.

(b) Compute the arc length parameter $s(\theta) = \int_0^\theta \|\alpha'(t)\| dt$.

(c) Find $\lim_{\theta \to +\infty} s(\theta)$.

2. Let $\alpha : [0, L] \to \mathbb{R}^2$ be a smooth curve of the *unit* speed, $\alpha(t) = (x(t), y(t))$ and $\alpha_c(t) = \alpha(t) + cN(t)$ where $N(t) = (-y'(t), x'(t))$.

(a) Using the Frenet formulae (p19) to compute $N'(t)$ and $\alpha_c'(t)$.

(b) Is it true that the vectors $\alpha_c'(t)$ and $\alpha'(t)$ are parallel for all t and all c ? Justify your answer.

3. Let $\alpha(t) = (t, \frac{1+t}{t}, \frac{1-t^2}{t})$.

(a) Compute $x - y + z =$?

(b) Computer the torsion τ.

(c) Why is $\tau = 0$?

4. Show that the solution set of the equation

$$xz + yz + xy = c$$

is a regular surface for each value of the constant c, other than zero.

5. Take any Riemann metric of the form

$$g = e^{2\rho}(du^2 + dv^2)$$

where $\rho = \rho(u, v)$ is a smooth function of u and v.

(a) Compute the connection ∇ of g, ($\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}$, $\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial v}$, etc.)

b) Compute the Gaussian curvature of the metric g, (Hint, Use the formulae on p237, Exercise 1-2).

6. The crew of the Enterprise live in a world they believe to be Euclidean. We know it is not and the physical metric of their world is $g = e^{2\rho}(du^2 + dv^2)$ with $e^{2\rho} = \frac{1}{(1-r^2)^2}$ where $r^2 = u^2 + v^2$. Use Problem 5 b) to compute the Gaussian curvature of their world.

7. As a result of a celestial storm, the metric of the Enterprise world is changed utterly. Assume that the new metric $\tilde{g}$ has Gaussian curvature $\tilde{K} < 0$ everywhere. Let two space ships leave the same point in different directions, each traveling along a geodesic of $\tilde{g}$. Show that the two ships will never meet again.

(Hint: Apply Gauss-Bonnet formula (p. 268-269) to a geodesic "di-angle" in a surface with $K < 0$).

8. The spaceship Enterprise moves in the disk $D = \{(x, y)|x^2 + y^2 < 1\}$. The time it takes to travel along the curve $\alpha : [a, b] \to D$ is $T = \int_a^b \frac{\sqrt{(x'(t))^2+(y'(t))^2}}{1-[(x'(t))^2+(y'(t))^2]} dt$. If $\alpha(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t))$ show that T is greater than or equal to the time it takes to go from $(r(a), 0)$ to $(r(b), 0)$ along the x-axis. Deduce that the spaceship will never reach $\partial D = \{(x, y)|x^2 + y^2 = 1\}$ in finite time no matter what path it takes.

9. Let $\Psi(x, y) = (u(x, y), v(x, y)) = (e^x \cos y, e^x \sin y)$, $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$.

a) Compute the matrix of the differential $\Psi_*|_{(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$ for any point (x, y).

b) Is the map Ψ conformal ? Justify your answer.