Mathematics 468
Homework 1 solutions

1. Is the set

$$\{(p,q) \in \mathbf{R}^2 | p,q \in \mathbf{Q}\}$$

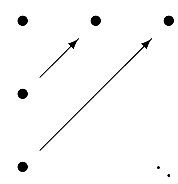
countable?

ANSWER: Yes. Since the set of rationals $\mathbf{Q}$ is countable, you can list them: $p_1, p_2, p_3, \ldots$. Now you can write down all of the ordered pairs of rationals in an array:

$$\begin{array}{ccccc}
(p_1,p_1) & (p_1,p_2) & (p_1,p_3) & (p_1,p_4) & \ldots \\
(p_2,p_1) & (p_2,p_2) & (p_2,p_3) & (p_2,p_4) & \ldots \\
(p_3,p_1) & (p_3,p_2) & (p_3,p_3) & (p_3,p_4) & \ldots \\
(p_4,p_1) & (p_4,p_2) & (p_4,p_3) & (p_4,p_4) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Now the set of all of these ordered pairs is countable: order them according to this chart:

$$\begin{array}{ccccc}
1 & 3 & 6 & 10 & \ldots \\
2 & 5 & 9 & 14 & \ldots \\
4 & 8 & 13 & 19 & \ldots \\
7 & 12 & 18 & 25 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$



2. If A_1, A_2, A_3, $\ldots$ are each countable, is their union?

ANSWER: Yes, for pretty much the same reason as in the first problem. Since each A_j is countable, you can list its elements:

$$A_j = \{a_{1j}, a_{2j}, a_{3j}, \ldots\}.$$

Now the union $\bigcup_{j=1}^{\infty} A_j$ consists of the elements in the following array (I've put the elements of A_j in the jth column):

$$\begin{array}{cccc}
a_{11} & a_{12} & a_{13} & \ldots \\
a_{21} & a_{22} & a_{23} & \ldots \\
a_{31} & a_{32} & a_{33} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Now cross out any repetitions, and count the remaining elements as in the first problem.

3. What is the smallest closed subset of $\mathbf{R}$ which contains $\mathbf{Q}$?

ANSWER: $\mathbf{R}$ itself is the smallest closed subset which contains $\mathbf{Q}$. Suppose that A is a subset of $\mathbf{R}$ which contains $\mathbf{Q}$; I claim that if $A \neq \mathbf{R}$, then A cannot be closed.

Since $A \neq \mathbf{R}$, then there is some point $x \in A^c$. For every $\varepsilon > 0$, the ball $B_\varepsilon(x)$ must contain at least one rational number (in fact, it contains countably many rationals, but that's not important for this problem). Therefore, this ball does not lie completely inside of A^c, and therefore A^c is not open. By the definition of "closed," A is not closed.

So if A is any proper subset of $\mathbf{R}$ which contains $\mathbf{Q}$, then A is not closed; hence $\mathbf{R}$ is the only closed subset of $\mathbf{R}$ which contains $\mathbf{Q}$.