Mathematics 468
Homework 2 solutions

1. Prove that in $\mathbf{R}^n$, the only sets which are both open and closed are the empty set and all of $\mathbf{R}^n$. (If you can't figure this out in general, try to do it when $n = 1$.)

ANSWER: I'll start with the $n = 1$ case, so suppose that U is a nonempty open subset of $\mathbf{R}^1$, and assume that its complement is nonempty; I will show that U cannot be closed. Since U is nonempty, I can pick a point x in U. Let A denote the complement of U, and pick a point z in A. I'll assume that $x < z$ (if $x > z$, then the same argument works, with slightly different notation). Since U is open, there is a y, $x < y < z$, so that the interval (x, y) is completely contained in U. In fact, there are many such y's. Let y_0 be the "largest such y": let

$$y_0 = \sup\{y \in (x, z) \,|\, (x, y) \subseteq U\}.$$

Then every neighborhood of y_0 intersects A, and every neighborhood of y_0 intersects U. If y_0 is in U, then U is not open; therefore, y_0 must be in A. But then A is not open, and so U is not closed. This is what I wanted to show. (This argument shows, in fact, that any interval—(a, b), $(a, b]$, $[a, b]$, or $[a, b)$—is connected.)

Now I'll do the general case. The setup is the same: let U be a nonempty open set, and assume that its complement $A = U^c$ is also nonempty. Pick points $x \in U$ and $z \in A$, and consider the straight line segment L connecting them. This line segment is connected, by the $n = 1$ case. Note that neither $U \cap L$ nor $A \cap L$ is empty. If U is both open and closed, then $U \cap L$ is both open and closed as a subset of L. Since L is connected, this can't happen; hence $\mathbf{R}^n$ has no proper subsets which are both open and closed.

2. If U_1, U_2, $\ldots$, U_n are all open, is their intersection? Prove or give a counterexample.

ANSWER: Yes, the intersection is open. This follows by induction on n. When $n = 1$, the intersection of one open set U_1 with itself is just U_1, which is open by assumption. (If you want a slightly less trivial case, then when $n = 2$, we proved in class that the intersection of two open sets is open.) Now, assume that the intersection of any $n - 1$ open sets is open, and consider

$$U_1 \cap U_2 \cap \ldots \cap U_{n-1} \cap U_n.$$

I can rewrite this as $(U_1 \cap \ldots \cap U_{n-1}) \cap U_n$. By the inductive hypothesis, the set $U_1 \cap \ldots \cap U_{n-1}$ is open, so by the $n = 2$ case (which we did in class), the intersection of this with U_n is still open.

3. If U_1, U_2, $\ldots$, U_n, $\ldots$ are all open, is their intersection? (The difference between this problem and the previous one is that in this case there are infinitely many sets U_i; in problem 2 there were only finitely many.)

ANSWER: No, the intersection doesn't have to be open. For example, if I let U_n be the open interval $(0, 1 + \frac{1}{n})$, then

$$\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} (0, 1 + \frac{1}{n}) = (0, 1].$$

The set $(0, 1]$ is certainly not open.

4. Suppose $f : \mathbf{R}^n \to \mathbf{R}^k$ is a function. What does it mean for f to be continuous? Try to explain it in words; also try to give a precise mathematical definition.

ANSWER: Both parts of this have several correct answers. For the precise mathematical definition, the usual one is that f is continuous if for every $x \in \mathbf{R}^n$ and every $\varepsilon > 0$, there is a $\delta > 0$ so that whenever $|y - x| < \delta$, then $|f(y) - f(x)| < \varepsilon$. (A less common, but equally useful, definition is that f is continuous if the preimage of every open set in $\mathbf{R}^k$ is open in $\mathbf{R}^n$.)