Mathematics 468
Homework 3 solutions

Given a subset A of a topological space X, let A' be the set of limit points of A. Let $\overline{A}$ denote the closure of A: let $\overline{A} = A \cup A'$.

1. Show that $\overline{A}$ is closed.

Solution: By a result from class, if I can show that $\overline{A}$ contains all of its limit points, then I can conclude that $\overline{A}$ is closed. Let y be a limit point of $\overline{A}$. I claim that every neighborhood of y intersects A, so that y is contained either in A or in A'. If $y \in A$, then there is nothing more to show, so I'll assume that $y \notin A$, and I'll show that every neighborhood of y intersects A. Let N be a neighborhood of y. If N contains a point of A, then that's good. Otherwise, N must contain a point x of A'. Since x is a limit point of A, then every neighborhood of x must contain a point of A; in particular, N is a neighborhood of x, and so must contain a point of A. Hence every neighborhood N of y contains a point of A, and hence y is a limit point of A, and hence y is in $\overline{A}$. Therefore $\overline{A}$ contains all of its limit points, so it is closed.

2. Show that $\overline{A}$ is the smallest closed subset of X containing A. In other words, if C is a closed set containing A, show that C contains $\overline{A}$ also. (Equivalently, show that $\overline{A}$ is the intersection of all closed sets containing A.)

Solution: Suppose that C is a closed set containing A. I want to show that C contains $\overline{A} = A \cup A'$; in other words, I want to show that C contains both A and A'. By assumption, C contains A, so given y in A' but not in A, I want to show that y is in C. Every neighborhood of y intersects A in some point other than y, and since $A \subseteq C$, then every neighborhood of y intersects C in some point other than y. Since C is closed, then C contains all of its limit points, so y must be contained in C.