Exam 1
March 4, 1998

General instructions: When I say "X is a topological space," I mean that X is a naïve topological space: X is a subset of $\mathbf{R}^n$ for some n.

Definitions.

(5) 1. Let X be a topological space and let A be a subset of X. Define what it means for a point y of X to be a *limit point* of A.

(5) 2. Define the term *connected*, as applied to topological spaces.

(5) 3. Let X and Y be topological spaces, and let $f : X \to Y$ be a function. What does it mean for f to be *continuous*?

True or false: For the next two problems, tell me whether each statement is true or false. If it's true, give a *brief* reason why. If it's false, give a brief reason or a counterexample.

(10) 4. The circle S^1, defined by

$$S^1 = \{(x, y) \in \mathbf{R}^2 \ : \ x^2 + y^2 = 1\},$$

is homeomorphic to the open interval $(0, 3)$.

(10) 5. Let $f : X \to Y$ be a function between topological spaces X and Y. Then $f^{-1}(Y) = X$.

Types of sets: For the next question, you may give reasons if you want, but they are optional. (If you get the answers right without any justification, you *will* get full credit.)

6. Let X be the topological space

$$X = [0, 1] \cup (2, 3] \cup \{4\} \cup \Big((6, 7) \cap \mathbf{Q}\Big),$$

viewed as a subset of $\mathbf{R}$. Let $A = (2, 3]$. Let $B = [0, \sqrt{40}) \cap X$. Let $C = (6, 7) \cap \mathbf{Q}$.

(5) (a) Is A open as a subset of X?

(5) (b) Is A compact?

(5) (c) Is A connected?

(5) (d) Is B open as a subset of X?

(5) (e) Is B closed as a subset of X?

(5) (f) Is B connected?

(5) (g) Is C open as a subset of X?

(5) (h) Is C open as a subset of $\mathbf{R}$?

(5) (i) Is 4 a limit point of X?

Theory: Do *two* of the following four problems.

(10) 7. Let X be a topological space and let A be a subset of X. Show that if A contains all of its limit points, then A is closed.

(10) 8. Let X be a topological space and let A be a subset of X. Show that if A is closed, then A contains all of its limit points.

(10) 9. Prove that the interval $[0, 1]$ is connected.

(10) 10. If A and B are countable sets, show that $A \cup B$ is countable, also. Then use this to explain why the set of irrational numbers is uncountable.