Mathematics 468 J. Palmieri

Final Exam

Due at noon, Friday, May 8 in 219 CCMB

General instructions: This is a closed book, closed notes exam. You may not use calculators, computers, etc. You may not discuss the exam with anyone except me. This exam is being conducted under the honor code. You may spend up to two hours working on the exam, preferably in one or two sittings.

If you need to reach me: E-mail: `John.H.Palmieri.2@nd.edu`. Office: 219 CCMB. Phone: 631-5352. I expect to be in my office most of this week (but you might want to call before you come by, just to be sure that I'm there).

Basic topology

(10) 1. Define what it means for a topological space to be *compact*. Give me an example of a compact space and of a non-compact space (no proofs required).

(10) 2. Define *homeomorphism*.

(25) 3. Some topological properties are preserved under images of continuous functions, some under preimages, and some under neither. Which properties fall into each category? For example, if X is connected, must $f(X)$ be connected for any continuous function $f : X \to Y$? What about $g^{-1}(X)$, for a continuous function $g : Z \to X$? For each topological property that you can think of, tell me whether it is preserved under images—if it is, just say so (no proof required); if it isn't, give an example. Do the same for preservation under preimages.

(15) 4. Do one of the following two parts (not both):

(a) Outline the proof of the Heine-Borel theorem: the unit interval $[0, 1]$ is compact. (Don't use the theorem about closed and bounded sets, because we used this property of $[0, 1]$ to prove that theorem.)

(b) Outline the proof of the theorem that a subset X of $\mathbf{R}^n$ is compact if and only if X is closed (as a subset of $\mathbf{R}^n$) and bounded. (You can use the Heine-Borel theorem here.)

Surfaces

(15) 5. In class we discussed two classification theorems for surfaces (i.e., two-dimensional manifolds); one theorem was for compact orientable surfaces, the other for arbitrary compact surfaces. State both of them.

Manifolds

(10) 6. Give me two examples of three-dimensional manifolds.

(10) 7. Let U be an open subset of $\mathbf{R}^k$. Define what it means for a function $f : U \to \mathbf{R}^n$ to be *smooth*.

8. Define $f : \mathbf{R}^3 \to \mathbf{R}^2$ by the formula $f(x, y, z) = (x^2 + y^2, x^2 + z^2)$. The function f is smooth (you don't need to prove this).

(5) (a) It turns out that the point $(1, 2) \in \mathbf{R}^2$ is a regular value of f. What conclusions can you draw from this? Draw some relevant pictures, if you can.

(5) (b) It turns out that the point $(0, 2)$ is a critical value of f. Is $f^{-1}(0, 2)$ a manifold? Give a brief justification.

(5) (c) Describe the subset $f^{-1}(1, 1)$ of $\mathbf{R}^3$. Is $(1, 1)$ a regular value or a critical value of f?

(5) (d) Find all of the regular values of f.

(15) 9. Prove that the hyperbola

$$H = \{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 = 1\}$$

is a one-dimensional manifold. For full credit, describe explicit parametrizations and coordinate functions. (If you use the theorem about regular values instead, you can get up to 10 points of partial credit.)

(10) 10. Let S^1 denote the unit circle in the complex plane, and consider the function $\varphi : \mathbf{R} \to S^1$ defined by $\varphi(t) = e^{it} = \cos t + i \sin t$. Compute the derivative of φ at $t = 0$, $d\varphi_0$, and interpret your answer as a linear function $d\varphi_0 : T_0\mathbf{R} \to T_1S^1$. (Draw pictures of the tangent spaces, describe the function $d\varphi_0$ between these tangent spaces, etc.)

11. Consider the function $g : S^1 \to S^1$ defined by $g(z) = z^3$.

(3) (a) What do you think dg_1, the derivative of g at $1 \in S^1$, is? Take your best guess.

(7) (b) Compute dg_1. (Hint: you might want to use the function φ from the previous problem for the parametrization.)