Final Exam

This exam is due on Thursday, May 6, at 5:30 pm, in my office.

This exam is being conducted under the honor code: you may not discuss it with anyone except me. You may use books and (your own) notes. If you get stuck on some problem, feel free to ask me for hints; the easiest way to do this is to come by my office. I should be in during most of the day Monday through Thursday, and available by e-mail other times. (Note that I'm giving another exam Thursday, 1:45-3:45, so I'm busy then)

The questions with only one part should be straight-forward, as should part (a) of each of the others. The later parts of the other problems are, for the most part, harder; some of them are quite hard. Please do four of the one-part problems and/or the (a) parts, and try the rest of two of the multi-part problems. (If you can't completely solve them, make as much progress as you can.)

Contact me if you have any questions.

1. Answer the first sentence of Exercise 6 from Section 10 (p. 67). Go into as much detail as you can.
2. Do Exercise 6 in Section 13 (p. 83).
3. Prove that the connected sum $P\#T$ is topologically equivalent to $P\#P\#P$.
4. (a) Let S be a subset of $\mathbf{R}^n$ for some n (if you need to, assume that n is 1 or 2, but try to work more generally if you can). The *closure* of S, written $\overline{S}$ or S^-, is the union of S with all of the near points of S. Show that $\overline{S}$ is closed for any S.

   Write S' for the complement of S. Together with closure, this gives two operations $S \mapsto S'$ and $S \mapsto S^-$ we can perform on a set, and we can iterate them and combine them in different ways. For example, starting with the open interval $S = (0,1)$ viewed as a subset of the line, the closure is $S^- = [0,1]$, the complement is $S' = (-\infty, 0] \cup [1, \infty)$, the complement of the closure is $S^{-\prime} = (-\infty, 0) \cup (1, \infty)$, and the closure of the complement is the same as the complement, and also the same as the closure of the complement of the closure: $S^{\prime -} = S' = S^{-\prime -} = (-\infty, 0] \cup [1, \infty)$. Note also that for any set S, $S^{--} = S^-$ (since S^- is closed), and $S'' = S$, so you don't get anything very interesting by doing either operation twice in a row—you have to alternate them.

   Kuratowski showed that starting with any set S, you can get at most 14 different sets by combining these two operations.

   (b) The set $S = (0,1)$ gave 4 different sets under these operations. Find a subset of $\mathbf{R}$ which gives 14 different sets, or if you can't manage that, as close to 14 as you can.

   (c) Prove Kuratowski's result. [Hint: first show that if A is the closure of an open set, then $A^{\prime - \prime -} = A$.]
5. (a) Let D be a cell. Show that the Brouwer fixed point theorem (p. 37 in the book) is equivalent to this: every continuous vector field v on D, for which the vector $v(P)$ lies entirely within D for each point $P \in D$, has a critical point.

   (b) Does the sphere have the fixed point property? What about the torus? The projective plane? If no, give an example. if yes, give a proof, or as much evidence as you can.
6. Let S be any compact connected surface, and suppose it has been triangulated. Let v be the number of vertices, e the number of edges, f the number of faces, and $\chi = v - e + f$ the Euler characteristic.

   (a) Show that $3f = 2e$, $e = 3(v - \chi)$, and $v \geq \dfrac{7 + \sqrt{49 - 24\chi}}{2}$.

   (b) Find lower bounds for the number of vertices for triangulations of the sphere, the torus, and the projective plane. What are the corresponding numbers of edges and faces?

   (c) Find examples of triangulations of these surfaces that have exactly these numbers of vertices, edges, and faces.

7. (a) Let θ denote the subset of the plane consisting of two vertices and three edges (no faces), arranged as in the first figure below. Compute the homology groups of θ.

   (b) Here is another subset of the plane: take a hexagon and glue it to another hexagon along an edge: see the second figure below. Compute its homology groups. (Just as in part (a), there are no faces here—just vertices and edges.)

   (c) According to what I said in class on Wednesday, $H_0(X) \cong \mathbf{Z}/2$ for any connected complex X, and $H_2(S) \cong \mathbf{Z}/2$ for any compact connected surface S. So if S is a compact connected surface, then $H_1(S)$ is the only "interesting" homology group. Describe as best you can these groups:

$$H_1(\underbrace{P\#\ldots\#P}_{m}), \quad H_1(\underbrace{T\#\ldots\#T}_{n})$$

   for all numbers $m, n \geq 1$.